Faculty of Mathematics and Physics,
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Goal-oriented a posteriori error estimates and where to find them

PANM 20

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Error estimation

A priori error estimates
$\|u - u_h\| \leq C(u)h^p$

A posteriori estimates
$\|u - u_h\| \leq \sum_{K\in\mathcal{T}_h} \eta_K(u_h)$

Standard approach
based on the formulation of the problem

Goal-oriented estimates
error of target functional
$J(u) - J(u_h) \leq \sum_{K\in\mathcal{T}_h} \eta_K(u_h)$

Content

Quantity of interest

Target functional

- we are not interested in u itself, but in the quantity of interest $J(u)$
- Examples:
  - mean heat flux through the part of boundary (Nusselt number)
  - regularized point value
  - drag and lift coefficient in aerodynamics
  - mean surface pressure of a body in an inviscid flow
- $J : V \to \mathbb{R}$ – target functional
- $J(u)$ may be linear or nonlinear (usually for nonlinear problems)

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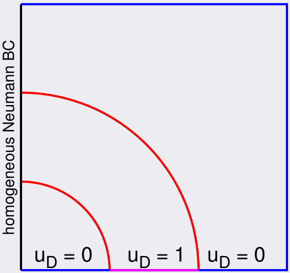
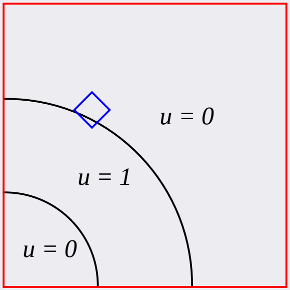
Problem setting

Primal problem: $-\varepsilon\Delta u + \nabla \cdot (\boldsymbol{b}u) = 0, \quad \boldsymbol{b} = (-x_2, x_1), \ \varepsilon = 10^{-6}$

Target functional $J(u) = \frac{1}{|E|} \int_E u \,\mathrm{d}x$

Boundary conditions and primal solution





Motivating experiment

Linear case with dominant convection

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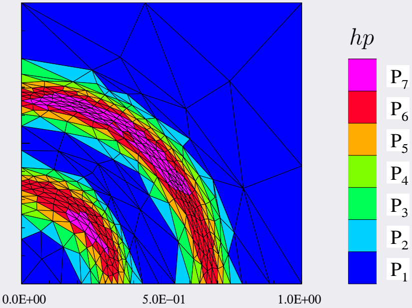
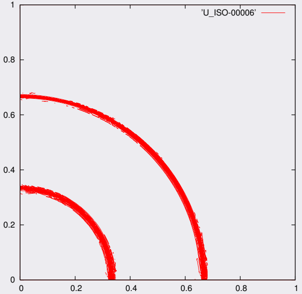
Adaptive refinement based on a standard estimates





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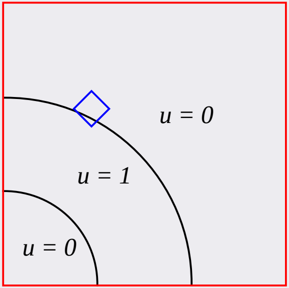
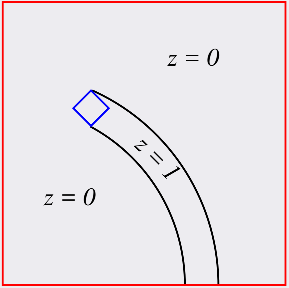
Adjoint problem: $-\varepsilon\Delta z - \boldsymbol{b} \cdot \nabla z = \chi_E$, where $E \subset \Omega$.

Primal and adjoint solutions





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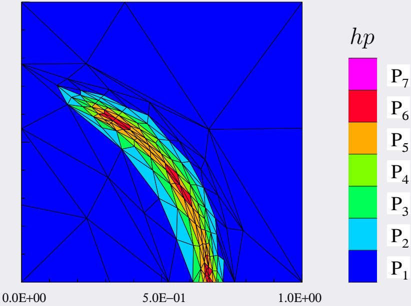
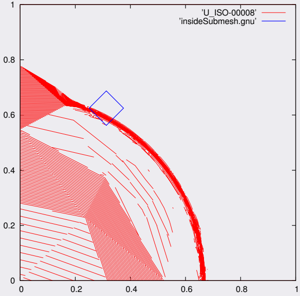
Adaptive refinement based on goal-oriented algorithm





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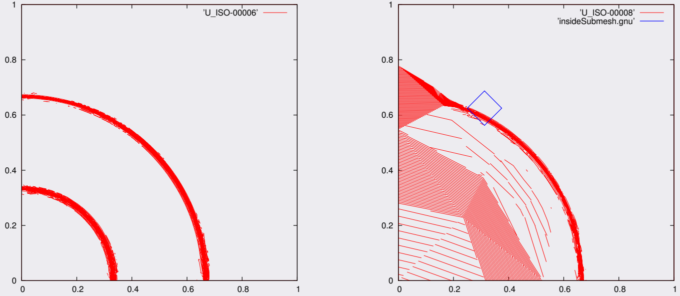
Comparison of the resulting solutions



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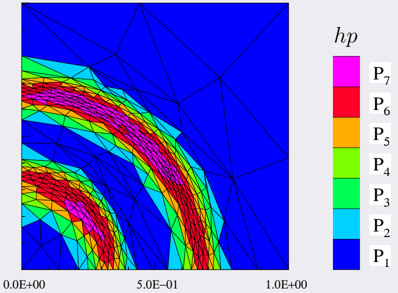
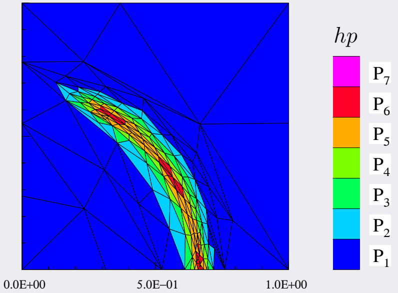
Comparison of resulting meshes





Poisson equation

Find a function u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Primal weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$a(u, \varphi) = \ell(\varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

where $a(u, \varphi) = (\nabla u, \nabla \varphi)_\Omega$ and $\ell(\varphi) = (f, \varphi)_\Omega$.

Adjoint weak formulation

Find $z \in H_0^1(\Omega)$ such that

$$a(\phi, z) = J(\phi) \quad \forall \phi \in H_0^1(\Omega)$$

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Find $u_h \in V_h := \{v \in C(\overline{\Omega});\ v\big|_k \in P^p(K), \forall K \in \mathcal{T}_h\}$ such that

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Error identity

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\begin{aligned}
J(u) - J(u_h) &= J(u - u_h) = a(u - u_h, z) \\
&= a(u - u_h, z - \varphi_h) \\
&= f(z - \varphi_h) - a(u_h, z - \varphi_h) \\
&=: r_h(u_h)(z - \varphi_h) \quad \forall \varphi_h \in V_h
\end{aligned}
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Estimate of the discretization error $J(u - u_h)$

- $J(u - u_h) = r_h(u_h)(z - \Pi z), \quad \Pi : V \to V_h$
- z has to be approximated numerically by z_h^+,
- z_h^+ must be in a richer space than V_h:

1. bigger problem: $z_h^+ \in V_h^+ : a(\varphi_h, z_h^+) = J(\varphi_h) \quad \forall \varphi_h \in V_h^+$,
2. same sized problem + reconstruction $z_h^+ = \mathscr{R}(z_h) \in V_h^+$

Computable goal-oriented error estimate

$$|J(u) - J(u_h)| \approx \eta^{\mathrm{I}} := \frac{1}{2}(\eta_{\mathrm{S}}(u_h, z_h) + \eta_{\mathrm{S}}^*(u_h, z_h))$$

where

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Computable goal-oriented error estimate

due to $z \approx z_h^+$, $u \approx u_h^+$ the estimate is not guaranteed upper bound

Focus of our work

- discontinuous Galerkin method
- adjoint consistent discretization
- algebraic errors
- *hp*-anisotropic mesh adaptation

Discontinuous Galerkin discretization

Primal discrete problem

Find $u_h \in S_h^p := \{v \in L^2(\Omega);\ v\big|_k \in P^p(K),\ \forall K \in \mathcal{T}_h\}$ such that

$$a_h(u_h, \varphi_h) \;=\; a_{DG}(u_h, \varphi) + J_h^\sigma(u_h, \varphi) = (f, \varphi_h) \quad \forall \varphi_h \in S_h^p,$$

where

$$\begin{aligned} a_{DG}(u, \varphi) \;&:=\; \sum_K \int_K \nabla u \cdot \nabla \varphi \,\mathrm{d}x - \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma (\langle \nabla u \rangle \cdot \boldsymbol{n}[\![\varphi]\!] \\ &+\; \Theta \, \langle \nabla \varphi \rangle \cdot \boldsymbol{n}[\![u]\!]) \;\mathrm{d}S, \quad \Theta \in \{-1, 0, 1\} \\ J_h^\sigma(u, \varphi) \;&:=\; \sum_{\mathscr{F}_h^I} \int_\Gamma \sigma [\![u]\!][\![\varphi]\!] \;\mathrm{d}S, \quad \sigma\big|_\Gamma = \frac{C_W}{h_\Gamma}. \end{aligned}$$

Adjoint discrete problem

Find $z_h \in S_h^p$ such that

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where

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Consistency of the discretization

$$\begin{array}{ccc}
\textbf{Primal continuous problem} & & \textbf{Adjoint continuous problem} \\
a(u,\varphi) = \ell(\varphi) \quad \forall \varphi \in V & \longleftrightarrow & a(\phi, z) = J(\phi) \quad \forall \phi \in V \\
\updownarrow \text{consistency} & & \updownarrow \text{adjoint consistency} \\
\textbf{Primal discrete problem} & & \textbf{Adjoint discrete problem} \\
a_h(u_h,\varphi_h) = \ell_h(\varphi_h) \quad \forall \varphi_h \in S_h^p & \longleftrightarrow & a_h(\phi_h, z_h) = J_h(\phi_h) \quad \forall \phi_h \in S_h^p
\end{array}$$

Consistency of the discretization

Primal continuous problem

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consistency

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\begin{tikzcd}[column sep=large, row sep=huge]
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Consistency

- All three variants of DG discretization (SIPG,NIPG,IIPG) are consistent.
- Only the symmetric variant (SIPG) of DG discretization is adjoint consistent.
- Problematic term in the discretization of the diffusive term:

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla\varphi \cdot \nabla z \,\mathrm{d}x - \sum_{\Gamma \in \mathscr{F}_h^{ID}} \int_\Gamma \langle \nabla\varphi \rangle \cdot \boldsymbol{n}[\![z]\!] + \theta \langle \nabla z \rangle \cdot \boldsymbol{n}[\![\varphi]\!] \,\mathrm{d}S.$$

Poisson problem

$$-\Delta u = f \text{ in } \Omega = (0,1)^2, \qquad u\big|_{\partial\Omega} = 0,$$

with f such that: $\quad u = 16x(1-x)y(1-y)$.

Quantity of interest: $\quad J(u) = \int_\Omega fu \,\mathrm{d}x \quad \Longrightarrow \quad z = u$

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Poisson problem

$$-\Delta u = f \text{ in } \Omega = (0,1)^2, \qquad u\big|_{\partial\Omega} = 0,$$

with f such that: $\quad u = 16x(1-x)y(1-y).$

Quantity of interest: $\quad J(u) = \int_\Omega fu \, \mathrm{d}x \quad \Longrightarrow \quad z = u$

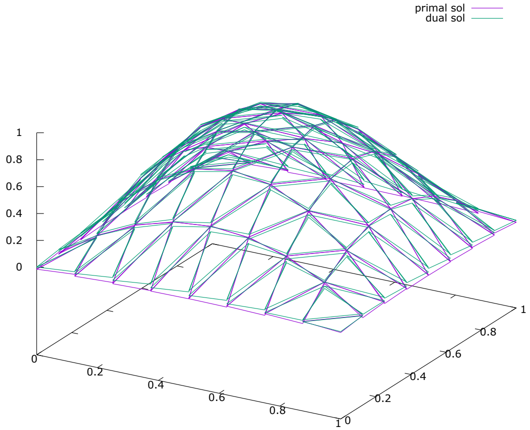

Figure: NIPG

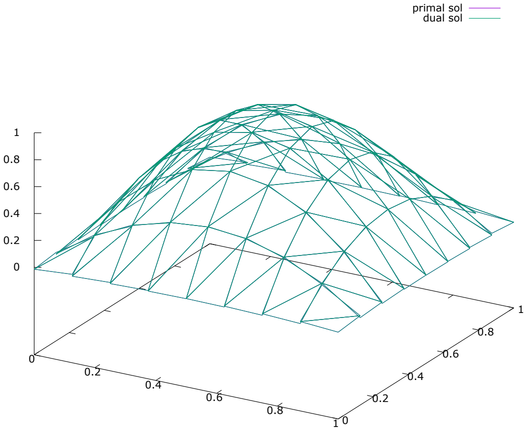

Figure: SIPG

Algebraic errors

Practically we do not have u_h, but only u_h^k, which is also suffering from algebraic errors. That leads to violation of the Galerkin orthogonality, i.e. $a_h(u - u_h^k, \varphi_h) \neq 0, \quad \varphi_h \in S_h^p$.

Two ways of splitting the error

$$
\begin{aligned}
J(u - u_h^k) &= a(u - u_h^k, z) = a(u - u_h^k, z - z_h^k) + a(u - u_h^k, z_h^k) \\
&= \ell_h(z - z_h^k) - a(u_h^k, z - z_h^k) + \ell_h(z_h^k) - a(u_h^k, z_h^k) \\
&\approx \underbrace{r_h(u_h^k)(z_h^+ - z_h^k)}_{\text{disc. error}} + \underbrace{r_h(u_h^k)(z_h^k)}_{\text{alg. error}}.
\end{aligned}
$$

$$
J(u - u_h^k) = J(u - u_h) + J(u_h - u_h^k)
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&\approx \underbrace{r_h(u_h^k)(z_h^+ - z_h^k)}_{\text{disc. error}} + \underbrace{r_h(u_h^k)(z_h^k)}_{\text{alg. error}} . \\
J(u - u_h^k) &= J(u - u_h) + J(u_h - u_h^k)
\end{aligned}
$$

Algebraic representation of the discrete problem

$$a_h(u_h, \varphi_h) = \ell_h(\varphi_h), \quad a_h(\psi_h, z_h) = J(\psi_h) \quad \forall \varphi_h, \psi_h \in S_h^p$$

$$\Downarrow$$

$$\mathbb{A}x = b \qquad \mathbb{A}^{\mathrm{T}}y = c$$

Biconjugate gradient method (BiCG)

- BiCG is a Krylov subspace method using short recurrences for solving altogether both systems $\mathbb{A}x = b$, $\mathbb{A}^{\mathrm{T}}y = c$
- In each iteration updates approximations x_k, y_k using coefficients α_k, β_k and vectors r_k, s_k, p_k, q_k.

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BiCG and goal-oriented estimates

Estimate of the algebraic error

$$J(u_h) = c^{\mathrm{T}} \mathbb{A}^{-1} b = \xi^P + \xi^B_k + s_k^{\mathrm{T}} \mathbb{A}^{-1} r_k,$$

where $\xi^P = c^{\mathrm{T}} x_0 + y_0^{\mathrm{T}} r_0$, $\xi^B_k = \sum_{i=0}^{k-1} \alpha_i s_i^{\mathrm{T}} r_i$. Then

$$J(u_h) - J(u_h^k) = \xi^B_{k+\nu} - \xi^B_k + s_{k+\nu}^{\mathrm{T}} \mathbb{A}^{-1} r_{k+\nu} \approx \xi^B_{k+\nu} - \xi^B_k$$

σ-stopping criterion [Dolejší and Tichý, 2020]

$\max(\sigma_{A,k}, \sigma^*_{A,k}) \leq c_A \omega$ where $\omega > 0$ is given tolerance and

$$\sigma_{A,k} := |\xi^B_{k+\nu} - \xi^B_k| + |y_k^{\mathrm{T}} r_k| \approx |J(u_h - u_h^k)| + |r_h(u_h^k)(z_h^k)|,$$
$$\sigma^*_{A,k} := |\xi^B_{k+\nu} - \xi^B_k| + |s_k^{\mathrm{T}} x_k| \approx |\ell_h(z_h - z_h^k)| + |r_h^*(z_h^k)(u_h^k)|$$

BiCG and goal-oriented estimates

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Our goal

Construct an anisotropic *hp*-mesh such that

- the error estimate η^{I} is under the given tolerance
- the number of Degrees of Freedom (DoF) is minimal

hp-anisotropic mesh adaptation

We optimize

1. size of each triangle K (adaptive)
2. shape of each triangle (anisotropic)
3. local polynomial approximation degree (*hp*-)

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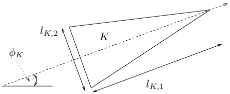
Anisotropic mesh adaptation

Each element of the mesh should have optimal size and shape.
Anisotropy of triangle is given by the triplet $\{\lambda_K, \sigma_K, \phi_K\}$:

- size $\lambda_K = \sqrt{l_{K,1} l_{K,2}}$,
- ratio $\sigma_K = \sqrt{l_{K,2}/l_{K,1}}$,
- orientation $\phi_K \in [0, \pi)$



hp-mesh adaptation

Each element K has its own pol. degree approximation p_K.

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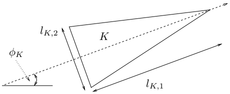

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Each element K has its own pol. degree approximation p_K.

Goal-oriented error estimates in residual form

$$|J(u) - J(u_h)| \approx \eta^{\mathrm{I}} \leq \sum_{K \in \mathcal{T}_h} \eta_K^{\mathrm{II}}$$

where

$$\begin{aligned}
\eta_K^{\mathrm{II}} := \frac{1}{2} \Big(& R_{K,\mathrm{V}} \left\| z_h^+ - \Pi z_h^+ \right\|_K + R_{K,\mathrm{V}}^* \left\| u_h^+ - \Pi u_h^+ \right\|_K \\
& + R_{K,\mathrm{B}} \left\| z_h^+ - \Pi z_h^+ \right\|_{\partial K} + R_{K,\mathrm{B}}^* \left\| u_h^+ - \Pi u_h^+ \right\|_{\partial K} \\
& + R_{K,\mathrm{D}} \left\| \mathbb{A} \nabla (z_h^+ - \Pi z_h^+) \right\|_{\partial K} + R_{K,\mathrm{D}}^* \left\| \mathbb{A} \nabla (u_h^+ - \Pi u_h^+) \right\|_{\partial K} \Big),
\end{aligned}$$

$$r_{K,\mathrm{V}}(u_h) := f + \Delta u_h, \qquad R_{K,\mathrm{V}} = \| r_{K,\mathrm{V}}(u_h) \|_K$$

$$r_{K,\mathrm{B}}(u_h) := \begin{cases} -\delta [\![u_h]\!] \boldsymbol{n} \cdot \boldsymbol{n}_K - \frac{1}{2} [\![\nabla u_h]\!] \cdot \boldsymbol{n} & \text{on } \partial K \setminus \partial \Omega, \\ \delta u_h & \text{on } \partial K \cap \partial \Omega, \end{cases} \qquad R_{K,\mathrm{B}} = \| r_{K,\mathrm{B}}(u_h) \|_{\partial K}$$

$$r_{K,\mathrm{D}}(u_h) := \begin{cases} \theta \frac{1}{2} [\![u_h]\!] & \text{on } \partial K \setminus \partial \Omega, \\ \theta u_h & \text{on } \partial K \cap \partial \Omega. \end{cases} \qquad R_{K,\mathrm{D}} = \| r_{K,\mathrm{D}}(u_h) \|_{\partial K}$$

Anisotropic goal-oriented error estimates

Using the anisotropy of the triangle $\{\lambda_K,\ \sigma_K,\ \phi_K\}$ and the anisotropy of $(u_h^+ - \Pi u_h^+)|_K$ denoted by $\{A_u,\ \rho_u,\ \varphi_u\}$ we estimate

$$\left\| u_h^+ - \Pi u_h^+ \right\|_K^2 \leq \frac{A_u^2 \lambda_K^{2(p_K+2)}}{2p_K+4} \boldsymbol{G}(p_K+1, p_K+1, \rho_u, \varphi_u; \sigma_K, \phi_K) =: \theta_{K,\mathrm{V}}^2,$$

etc. and then

$$\begin{aligned} \eta_K^{\mathrm{II}} \leq \eta_K^{\mathrm{III}} := \frac{1}{2}(&R_{K,\mathrm{V}}\theta_{K,\mathrm{V}}^* + R_{K,\mathrm{B}}\theta_{K,\mathrm{B}}^* + R_{K,\mathrm{D}}\theta_{K,\mathrm{D}}^* \\ &+ R_{K,\mathrm{V}}^*\theta_{K,\mathrm{V}} + R_{K,\mathrm{B}}^*\theta_{K,\mathrm{B}} + R_{K,\mathrm{D}}^*\theta_{K,\mathrm{D}}) \end{aligned}$$

Solve primal and adjoint problems

compute $u_h \in S_h^p$ and $z_h \in S_h^p$

Reconstruct discrete solutions

compute $u_h^+ \in S_h^{p+1}$ and $z_h^+ \in S_h^{p+1}$

Evaluate η^{I} and η_K^{I}

IF $\eta^{\mathrm{I}} < TOL$ then STOP;

Optimize element shapes

Foreach $K \in \mathcal{T}_h$, based on η_K^{III}

1. set better element size $\tilde{\lambda}_K$
2. set better element shape $\tilde{\phi}_K$, $\tilde{\sigma}_K$
3. set better element pol. degree $\tilde{p}_K$

Generate new mesh

from the set of quintets $\{x_K, \tilde{\lambda}_K, \tilde{\sigma}_K, \tilde{\phi}_K, \tilde{p}_K\}$,

generate a new *hp*-mesh $\mathscr{T}_{h,p}^{n+1} = \{\mathcal{T}_h^{(n+1)}, p^{n+1}\}$

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Anisotropic *hp*-mesh adaptation algorithm

Solve primal and adjoint problems

compute $u_h \in S_h^p$ and $z_h \in S_h^p$

Reconstruct discrete solutions

compute $u_h^+ \in S_h^{p+1}$ and $z_h^+ \in S_h^{p+1}$

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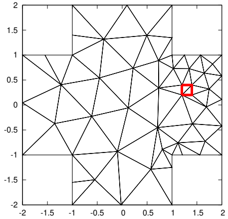
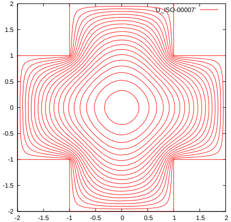
generate a new *hp*-mesh $\mathscr{T}_{h,p}^{n+1} = \{\mathcal{T}_h^{(n+1)}, p^{n+1}\}$

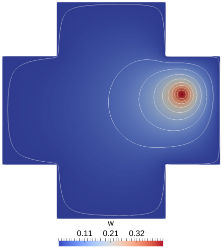
Problem setting

$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

in the cross shaped domain $\Omega = (-2,2) \times (-1,1) \cup (-1,1) \times (-2,2)$.







Poisson problem on cross-shaped domain

Algebraic error decrease

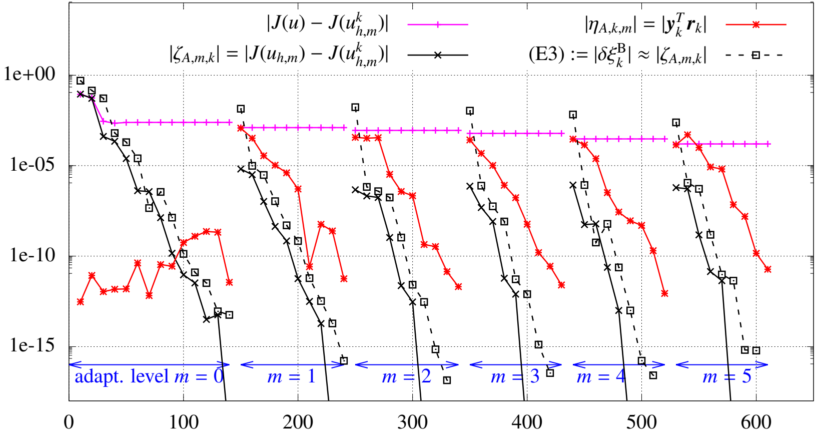


Figure: Adaptive computations for mesh adaptation steps $m = 0, \ldots, 5$, primal quantities.

Poisson problem on cross-shaped domain
Algebraic error decrease

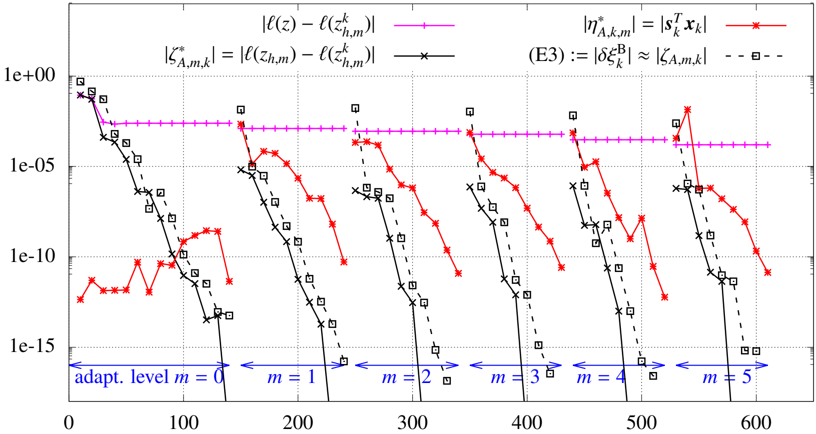


Figure: Adaptive computations for mesh adaptation steps $m = 0, \ldots, 5$, adjoint quantities.

Poisson problem on cross-shaped domain

Adaptive computation

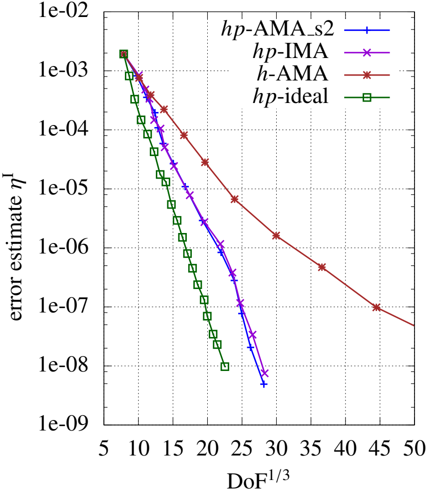


Figure: Convergence of the error estimate $\eta^{\mathrm{I}} = \eta^{\mathrm{I}}(u_h^{(n)}, z_h^{(n)})$ w. r. t. DoF for four adaptation strategies.

Poisson problem on cross-shaped domain

Adaptive computation

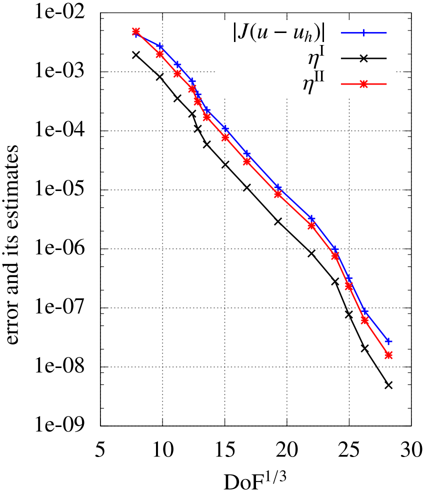


Figure: Comparison of the error estimates η^{I} and η^{II} with the actual error $J(u-u_h)$ for the *hp*-AMA method

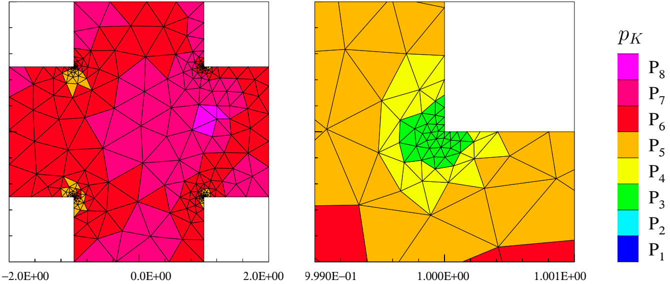


Figure: The final *hp*-mesh obtained by the *hp*-AMA method, total view (left) and a 1000× zoom of the corner singularity at x = (1, 1) (right).

Mixed hyperbolic elliptic problem
Numerical experiment

Problem setting

$$-\nabla \cdot (\varepsilon \nabla u) + \nabla \cdot (\boldsymbol{b} u) + c u = 0$$

where

$$\varepsilon = \frac{\delta}{2}\left(1 - \tanh\left(\frac{(r-\frac{1}{4})(r+\frac{1}{4})}{\gamma}\right)\right), \qquad r = \sqrt{(x-1/2)^2 + (y-1/2)^2}$$

$$\boldsymbol{b} = (2y^2 - 4x + 1, y + 1), \qquad c = -\nabla \cdot \boldsymbol{b}.$$

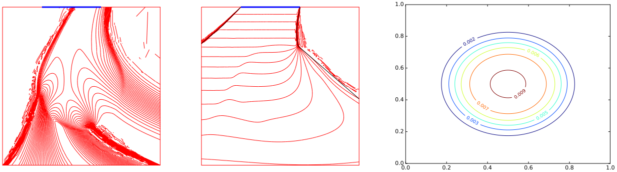


Figure: Primal (left) and adjoint (center) solutions and the function ε (right).

Problem setting

$$-\nabla \cdot (\varepsilon \nabla u) + \nabla \cdot (\boldsymbol{b} u) + cu = 0$$

Boundary conditions:

- $u_D = \begin{cases} 1 & \text{if } x = 0 \text{ and } 0 < y \leq 1, \\ \sin^2(\pi x) & \text{if } 0 \leq x \leq 1 \text{ and } y = 0, \\ e^{-50y^4} & \text{if } x = 1 \text{ and } 0 < y \leq 1, \end{cases}$
- homogeneous Neumann if $0 \leq x \leq 1$ and $y = 1$.

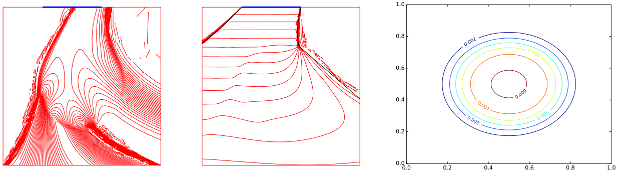


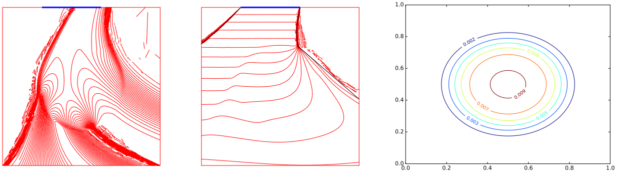
Figure: Primal (left) and adjoint (center) solutions and the function ε (right).

Problem setting

$$-\nabla \cdot (\varepsilon \nabla u) + \nabla \cdot (\boldsymbol{b} u) + cu = 0$$

Target functional:

$$J(u) = \int_{0.25}^{0.625} u(x, 1) \, \mathrm{d}x$$



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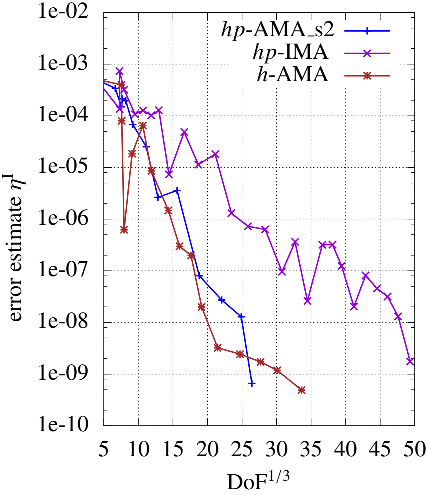

Figure: Convergence of the error estimate $\eta^{\mathrm{I}} = \eta^{\mathrm{I}}(u_h^{(n)}, z_h^{(n)})$ w. r. t. DoF for three adaptation strategies.

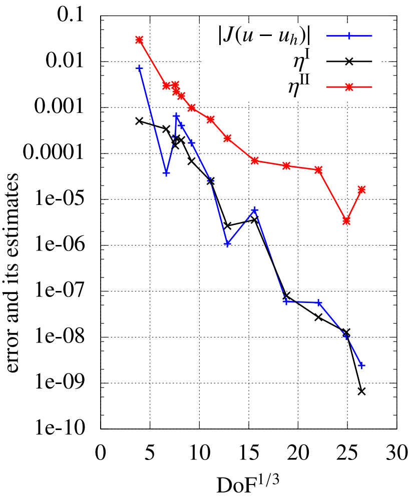

Figure: Comparison of the error estimates η^{I} and η^{II} with the actual error $J(u-u_h)$ for the *hp*-AMA method

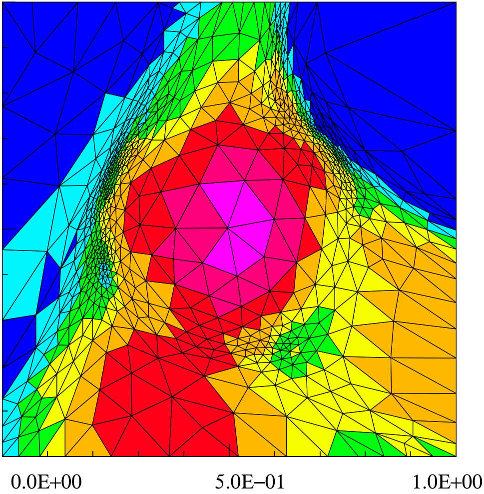

Figure: The final *hp*-mesh obtained by the *hp*-AMA method.

Nonlinear problems

Abstract nonlinear problem

Determine the value of the goal-functional

$$J(u) = \int_\Omega j_\Omega(u)\,\mathrm{d}x + \int_{\partial\Omega} j_\Gamma(u)\ \mathrm{d}S,$$

given that u solves

$$\mathcal{A}(u) = 0 \text{ in } \Omega, \qquad \mathcal{B}(u) = 0 \text{ on } \partial\Omega.$$

Abstract discrete problem

Find $u_h \in V_h$ such that

$$a_h(u_h; \varphi_h) = 0 \qquad \forall \varphi_h \in V_h.$$

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Abstract adjoint problem

Fréchet derivative

A function $f : V \to W$ is called *Fréchet differentiable* at $x \in V$ if there exist a continuous linear operator $f'[x] : V \to W$ such that

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - f'[x](h)\|_W}{\|h\|_V} = 0.$$

Adjoint problem

Find a function $z \in V$ such that

$$(\mathcal{A}'[u])^* z = j'_\Omega[u] \quad \text{in } \Omega, \qquad (\mathcal{B}'[u])^* z = j'_\Gamma[u] \quad \text{on } \partial\Omega,$$

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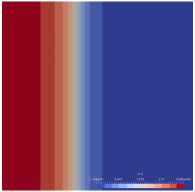
Motivating experiment

Problem setting

$$
\begin{aligned}
-\varepsilon \nabla \cdot (|u|^{\gamma} \nabla u) + \nabla(\boldsymbol{b} u) &= f \text{ in } \Omega = (0,1)^2, \\
u &= g_D \text{ on } \Gamma_D = \{(0, x_2) \in \partial\Omega\}, \\
\nabla \cdot u &= g_N \text{ in } \Gamma_N = \partial\Omega \backslash \Gamma_D,
\end{aligned}
$$

where $\varepsilon = 10^{-2}$, $\boldsymbol{b} = (1,0)^{\mathrm{T}}$, $\gamma \geq 0$.
Exact solution:

$$u = \arctan(-25(x_1 - 0.4)) + \frac{\pi}{2}$$



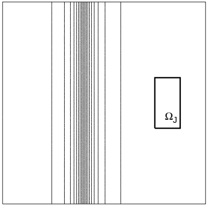
Problem setting

Target functional:

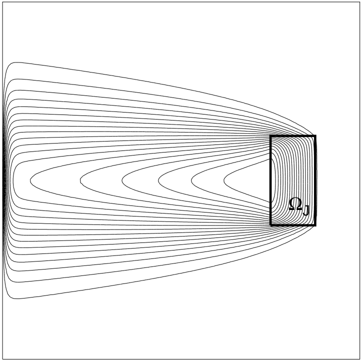
$$J(u) = \frac{1}{|\Omega_J|} \int_{\Omega_J} u \,\mathrm{d}x$$

Adjoint problem:

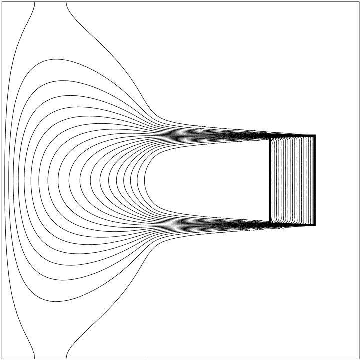
$$-\varepsilon \nabla \cdot (|u|^{\gamma} \nabla z + \gamma |u|^{\gamma - 1} \nabla u z) - \boldsymbol{b} \cdot \nabla z = \frac{1}{|\Omega_J|} \chi_{\Omega_J}.$$



Adjoint solution



(a) Linear case ($\gamma = 0$)



(b) Nonlinear case ($\gamma = 2$)

Discrete adjoint problem

Find a function $z_h \in V_h$ such that

$$a_h{}'[u_h](\psi_h, z_h) = J'[u_h](\psi_h) \qquad \forall \psi_h \in V_h.$$

Residuals

$$\begin{aligned} r_h(u_h)(\cdot) &= -A(u_h; \cdot), \\ r_h^*(z_h)(\cdot) &= J'[u_h](\cdot) - A'[u_h](\cdot, z_h) \end{aligned}$$

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Goal-oriented error estimate

Error identity for nonlinear problems

$$J(u) - J(u_h) = \frac{1}{2} r_h(u_h)(z - \psi_h) + \frac{1}{2} r_h^*(z_h)(u - \phi_h) + R_h^3 \qquad \forall \psi_h, \phi_h \in V_h,$$

where

$$\begin{aligned} r_h(u_h)(\cdot) &= -A(u_h; \cdot), \\ r_h^*(z_h)(\cdot) &= J'[u_h](\cdot) - A'[u_h](\cdot, z_h) \end{aligned}$$

and the remainder R_h^3 is cubic in the primal and adjoint errors

$$e = u - u_h, \quad e^* = z - z_h.$$

Euler equations

$$\sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = 0, \quad \boldsymbol{f}_s(\boldsymbol{w}) = \begin{pmatrix} \rho v_s \\ \rho v_1 v_s + \delta_{1s}\mathrm{p} \\ \vdots \\ \rho v_d v_s + \delta_{ds}\mathrm{p} \\ (E+\mathrm{p}) v_s \end{pmatrix}$$

with boundary conditions $\boldsymbol{v} \cdot n = 0$ on Γ_W and proper boundary conditions on Γ_{IO}, [Dolejí, Feistauer, 2015].

Quantity of interest - drag or lift

$$J(\boldsymbol{w}) = \int_{\Gamma_W} \mathrm{p}(\boldsymbol{w}) n \cdot \vartheta \, \mathrm{d}S, \quad \text{where } \mathrm{p} = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2),$$

$$\vartheta_d = \frac{1}{C_\infty}(\cos(\alpha), \sin(\alpha))^{\mathrm{T}} \quad \vartheta_l = \frac{1}{C_\infty}(-\sin(\alpha), \cos(\alpha))^{\mathrm{T}},$$

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Discrete primal problem

We say that a function $\boldsymbol{w}_h \in \boldsymbol{S}_{h,p}$ is the discrete solution of the Euler equations, if

$$a_h(\boldsymbol{w}_h, \varphi_h) = 0 \qquad \forall \varphi_h \in \boldsymbol{S}_{h,p},$$

where $a_h(\boldsymbol{w}, \varphi) = -\sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^d \boldsymbol{f}_s(\boldsymbol{w}) \cdot \frac{\partial \varphi}{\partial x_s} \,\mathrm{d}x + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbf{H}(\boldsymbol{w}^{(+)}, \boldsymbol{w}^{(-)}, n) \cdot \varphi \,\mathrm{d}S$

Discrete adjoint problem

Find $\boldsymbol{z}_h \in \boldsymbol{S}_{h,p}$: $\quad a_h'[\boldsymbol{w}_h](\varphi_h, \boldsymbol{z}_h) = J'[\boldsymbol{w}_h](\varphi_h) \qquad \forall \varphi_h \in \boldsymbol{S}_{h,p}$, where

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DG discretization

Discrete primal problem

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Discrete adjoint problem

Find $\boldsymbol{z}_h \in \boldsymbol{S}_{h,p}: \quad a_h'[\boldsymbol{w}_h](\varphi_h, \boldsymbol{z}_h) = J'[\boldsymbol{w}_h](\varphi_h) \qquad \forall \varphi_h \in \boldsymbol{S}_{h,p}$, where

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Main focus

- boundary conditions in the discrete formulation
- adjoint consistency (depends on the b.c., $J \approx J_h$)
- approximation of $a_h{}'[\boldsymbol{w}_h](\cdot,\cdot) \approx \boldsymbol{a}_h^{\mathrm{L}}(\boldsymbol{w}_h;\cdot,\cdot)$
  - for definition of the linearized discrete adjoint problem
  - for solving the nonlinear primal algebraic problem (Newton-like method)
- anisotropic *hp*-mesh adaptation

$$|J(u) - J(u_h)| \approx \eta^{\mathrm{I}} \leq \eta^{\mathrm{II}} \leq \sum_{K \in \mathcal{T}_h} \eta_K^{\mathrm{III}}$$

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Standard goal-oriented error estimates

$$|J(\boldsymbol{w}) - J(\boldsymbol{w}_h)| \approx \eta^{\mathrm{I}}$$
$$:= \frac{1}{2}|r_h(\boldsymbol{w}_h)(\boldsymbol{z}_h^+ - \Pi \boldsymbol{z}_h^+) + \frac{1}{2} r_h^*(\boldsymbol{w}_h, \boldsymbol{z}_h)(\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+)|,$$

where

$$r_h(\boldsymbol{w}_h)(\boldsymbol{z}_h^+ - \Pi \boldsymbol{z}_h^+) := -a_h(\boldsymbol{w}_h; \boldsymbol{z}_h^+ - \Pi \boldsymbol{z}_h^+),$$
$$r_h^*(\boldsymbol{w}_h, \boldsymbol{z}_h)(\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+) := J_h'[\boldsymbol{w}_h](\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+) - \boldsymbol{a}_h^{\mathrm{L}}[\boldsymbol{w}_h](\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+, \boldsymbol{z}_h)$$

Goal-oriented error estimates in residual form

$$|J(u) - J(u_h)| \approx \eta^{\mathrm{I}} \leq \sum_{K \in \mathcal{T}_h} \eta_K^{\mathrm{II}}$$

where

$$\eta_K^{\mathrm{II}}(\boldsymbol{w}_h, \boldsymbol{z}_h) = \frac{1}{2} \left(\sum_{i=1}^{m} R_{K,V}^{i} \left\| (\boldsymbol{z}_h^{+} - \Pi \boldsymbol{z}_h^{+})^{i} \right\|_{K} + R_{K,B}^{i} \left\| (\boldsymbol{z}_h^{+} - \Pi \boldsymbol{z}_h^{+})^{i} \right\|_{\partial K} \right.$$
$$\left. + R_{K,V}^{*,i} \left\| (\boldsymbol{w}_h^{+} - \Pi \boldsymbol{w}_h^{+})^{i} \right\|_{K} + R_{K,B}^{*,i} \left\| (\boldsymbol{w}_h^{+} - \Pi \boldsymbol{w}_h^{+})^{i} \right\|_{\partial K} \right).$$

$$R_{K,V}^{i} := \left\| \mathbf{R}_K^{i}(\boldsymbol{w}_h) \right\|_{K}, \quad R_{K,B}^{i} := \left\| \mathbf{r}_K^{i}(\boldsymbol{w}_h) \right\|_{\partial K}$$
$$R_{K,V}^{*,i} := \left\| \mathbf{R}_K^{*,i}(\boldsymbol{w}_h, \boldsymbol{z}_h) \right\|_{K}, \quad R_{K,B}^{*,i} := \left\| \mathbf{r}_K^{*,i}(\boldsymbol{w}_h, \boldsymbol{z}_h) \right\|_{\partial K}$$

Anisotropic goal-oriented error estimates

Using the anisotropy of the triangle $\{\lambda_K,\ \sigma_K,\ \phi_K\}$ and the anisotropy of $(\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+)|_K$ we estimate

$$\left\|(\boldsymbol{w}_h^+ - \Pi \boldsymbol{w}_h^+)^i\right\|_{\boldsymbol{K}}^2 \leq \underbrace{\frac{A_u^2 \lambda_K^{2(p_K+2)}}{2p_K+4} \boldsymbol{G}(p_K+1, p_K+1, \rho_u, \varphi_u; \sigma_K, \phi_K)}_{(\theta^i_{K,\mathrm{V}})^2}$$

etc. and then

$$\eta_K^{\mathrm{II}} \leq \eta_K^{\mathrm{III}} := \frac{1}{2}\left(\sum_{i=1}^{m} R^i_{K,V}\theta^{*,i}_{K,\mathrm{V}} + R^i_{K,B}\theta^{*,i}_{K,\mathrm{B}} + R^{*,i}_{K,V}\theta^i_{K,\mathrm{V}} + R^{*,i}_{K,B}\theta^i_{K,\mathrm{B}}\right).$$

Problem setting

- NACA0012 airfoil
- Mach number $M = 0.5$,
- angle of attack $\alpha = 0.0^\circ$,
- exact value of drag: $c_D = 0.0$.

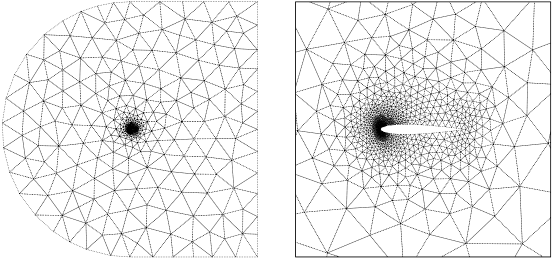


Figure: Initial computational mesh

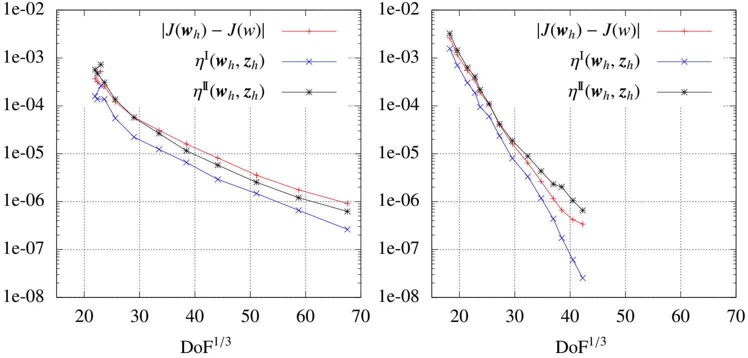


Figure: Decrease of the error and its estimates for anisotropic $p = 2$ and *hp*-mesh refinement for the drag coefficient.

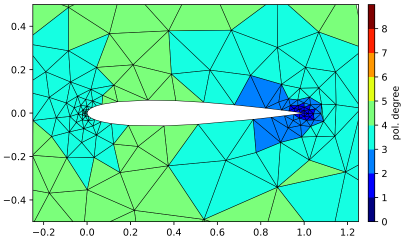
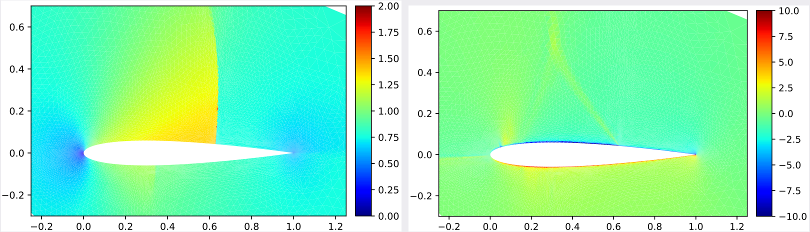


Figure: Refined *hp*-meshes 5th (left) 13th (right).

Problem setting

- NACA0012 airfoil
- Mach number $M = 0.8$,
- angle of attack $\alpha = 1.25^\circ$,
- target functional: c_D, c_L.

Drag coefficient



(a) First component of $\boldsymbol{w}_h$. (b) First component of $\boldsymbol{z}_h$.

Drag coefficient

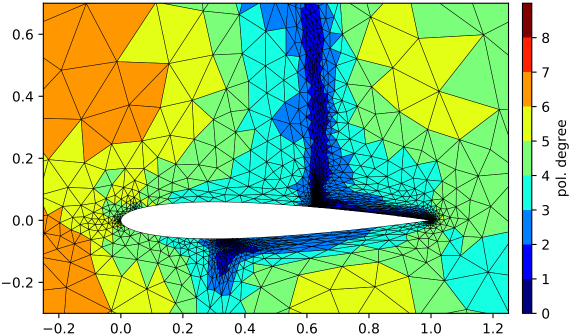


Figure: Final *hp*-mesh.

Drag coefficient

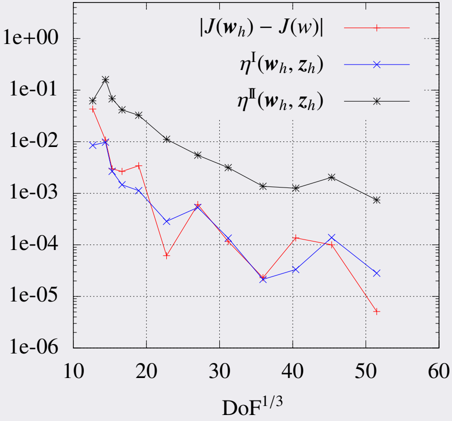


Figure: Error decrease.

Lift coefficient

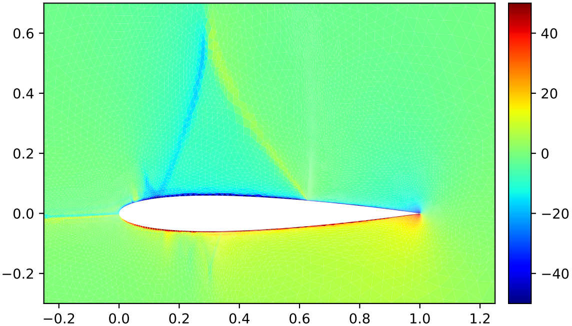
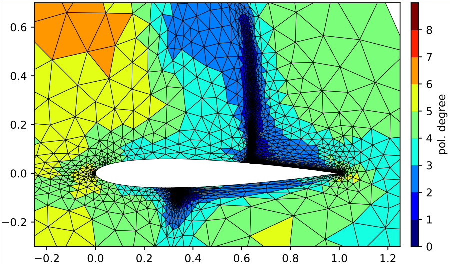


Figure: First component of the adjoint solution $\boldsymbol{z}_h$.

Lift coefficient



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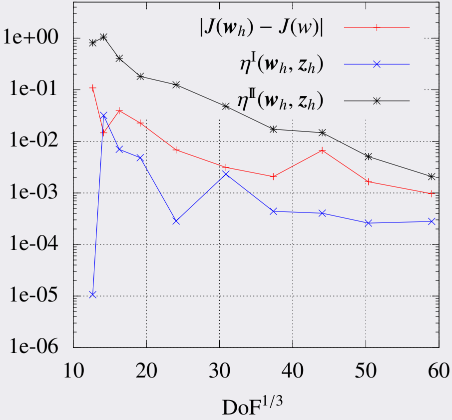

Figure: Error decrease.

Summary

- goal-oriented error estimates for linear and nonlinear equations
- adjoint consistent discretizations
- approximation of the adjoint solution z
- error estimates including algebraic errors
- adaptive refinement driven by the quantity of interest (*hp*-anisotropic)
- application to Euler equations

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Thank you for your attention!