Explicit time integration in finite element method for structural dynamic, wave propagation and contact-impact problems: A recent progress

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PANM, Hejnice, Czech Republic
June 23, 2020

Contents

1. **Motivation**

Accurate modelling of problems of structural dynamics, wave propagation and contact-impact tasks by advanced methods in the finite element method in dynamics.

Basic literature

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2. Governing equations for elastodynamic problems, Hamilton's principle in elastodynamic problems

Governing equations for elastodynamic problem

Strong form:

$$
\begin{aligned}
\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} \quad & \text{in} \quad \Omega \times [t^0, T] \\
\mathbf{u} = \hat{\mathbf{u}} \quad & \text{on} \quad \Gamma_D \times [t^0, T] \\
\mathbf{n}\cdot\boldsymbol{\sigma} = \hat{\mathbf{t}} \quad & \text{on} \quad \Gamma_N \times [t^0, T] \\
\mathbf{u}\left(\mathbf{x}, t^0\right) = \mathbf{u}_0\left(\mathbf{x}\right) \quad & \text{for} \quad \mathbf{x} \in \Omega \\
\dot{\mathbf{u}}\left(\mathbf{x}, t^0\right) = \dot{\mathbf{u}}_0\left(\mathbf{x}\right) \quad & \text{for} \quad \mathbf{x} \in \Omega
\end{aligned}
$$

the Hooke's law and the infinitesimal strain tensor:

$$
\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left[(\operatorname{grad}\boldsymbol{u})^{\mathsf{T}} + \operatorname{grad}\boldsymbol{u}\right]
$$

u_i - the component of displacement vector $\mathbf{u}(\mathbf{x}, t)$;
$\mathbf{x} \in \Omega$ - the position vector;
Ω - the domain of interest with the boundary Γ
σ_{ij} - the Cauchy stress tensor (symmetric tensor); ε_{kl} - the infinitesimal strain tensor;
ρ - mass density;
b_i - the component of volume (body) intensity vector $\mathbf{b}$;
n_i - the component of the outward normal vector $\mathbf{n}$ on Γ;
$\hat{u}_i$ - the component of prescribed boundary displacement vector $\mathbf{g}$;
$\hat{t}_i$ - the component of prescribed traction vector $\mathbf{t}$;
u_{0i} and $\dot{u}_{0i}$ - the components of the initial displacement and velocity fields.

Linear hyperbolic PDE system 15 unknown fields.

Hamilton's principle in dynamic problems

- the principle of stationary action of dynamic problems

$$\delta \int_{t_1}^{t_2} \left(\mathcal{T}(\dot{\mathbf{u}}) - (\mathcal{U}(\mathbf{u}) - \mathcal{W})\right) \, \mathsf{d}t = 0$$

$\mathcal{T}(\dot{\mathbf{u}})$ - Kinetic energy of the body

$$\mathcal{T}\left(\dot{\mathbf{u}}\right) = \int_{\Omega} \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, \mathrm{d}V$$

$\mathcal{U}(\mathbf{u})$ - Strain energy of the body

$$\mathcal{U}\left(\mathbf{u}\right) = \int_{\Omega} \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}V$$

$\mathcal{W}$ - Work of external forces on the body

$$\mathcal{W}\left(\mathbf{u}\right) = \int_{\Omega} \mathbf{u} \cdot \mathbf{b} \, \mathrm{d}V + \int_{\Gamma_N} \mathbf{u} \cdot \hat{\mathbf{t}} \, \mathrm{d}S$$

3. Finite element method in dynamics

FEM recapitulation

Approximation of displacement field via shape functions $\mathbf{N}$

$$\mathbf{u}^h = \mathbf{N}\,\mathbf{q},\ \delta\mathbf{u}^h = \mathbf{N}\,\delta\mathbf{q}$$

where $\mathbf{q}$ is vector of generalized nodal quantities (displacements/rotations, etc.).
Approximation of velocity and acceleration fields

$$\dot{\mathbf{u}}^h = \mathbf{N}\,\dot{\mathbf{q}} \qquad \ddot{\mathbf{u}}^h = \mathbf{N}\,\ddot{\mathbf{q}}$$

Infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u}^h,$$

where $\mathbf{D}$ is the differential operator. Then

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{N}\,\mathbf{q} = \mathbf{B}\,\mathbf{q}$$

where $\mathbf{B}$ is the strain-displacement matrix. In elasticity problems, stress is given as

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon}$$

where $\mathbf{E}$ is the elasticity matrix.

FEM recapitulation

Energy balance (principle of virtual work):

$$\int_\Omega \delta\mathbf{u}^T \varrho\ddot{\mathbf{u}} \, \mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_\Omega \delta\mathbf{u}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_N} \delta\mathbf{u}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Using discretization of kinematic quantities we have

$$\delta\mathbf{q}^T \left[\int_\Omega \varrho\mathbf{N}^T\mathbf{N}\ddot{\mathbf{q}} \, \mathrm{d}\Omega + \int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega - \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega - \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma \right] = \mathbf{0}.$$

The previous equation should be valid for an arbitrary $\delta\mathbf{q}$ respecting Dirichlet boundary conditions and, then the discretized equations of motion have the form

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}^{ext} - \mathbf{f}^{int}$$

FEM recapitulation

Discretized equations of motion:

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}^{ext} - \mathbf{f}^{int}$$

Consistent mass matrix:

$$\mathbf{M} = \int_{\Omega} \varrho \mathbf{N}^T \mathbf{N} \; \mathrm{d}\Omega$$

Vector of internal forces:

$$\mathbf{f}^{int} = \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \; \mathrm{d}\Omega$$

Vector of external forces:

$$\mathbf{f}^{ext} = \int_{\Omega} \mathbf{N}^T \mathbf{b} \; \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \; \mathrm{d}\Gamma$$

In impact-contact problems, equations of motion have the form

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}^{ext} - \mathbf{f}^{int} - \mathbf{f}^{contact}$$

FEM for linear problems

The continuous Galerkin-Bubnov approximation method.
Finite element approximation of the displacement field $\mathbf{u}$:

$$\mathbf{u}^h(\mathbf{x},t) = \sum_{I=1}^{NDOF} \mathbf{N}_I(\mathbf{x})\mathbf{u}_I(t), \qquad \delta\mathbf{u}^h(\mathbf{x},t) = \sum_{I=1}^{NDOF} \mathbf{N}_I(\mathbf{x})\delta\mathbf{u}_I(t)$$

where $\mathbf{u}_I$ are unknown nodal displacements.
Discrete equations of motion for linear elasticity problems:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}^{ext}$$

$+$ nodal Dirichlet boundary conditions.
Internal forces are given as

$$\mathbf{f}^{int} = \mathbf{K}\mathbf{u}$$

with the stiffness matrix defined as

$$\mathbf{K} = \int_\Omega \mathbf{B}^T\mathbf{E}\mathbf{B}\ \mathrm{d}\Omega$$

Damping

General nonlinear problems:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{f}^{int}(\mathbf{u}, \varepsilon, \dot{\varepsilon}) = \mathbf{f}^{ext}(t) \tag{1}$$

Damping matrix

$$\mathbf{D} = \int_\Omega d\mathbf{N}^T\mathbf{N}\ \mathrm{d}\Omega$$

where d is viscous damping parameter.

Rayleigh damping matrix:

$$\mathbf{D} = a\mathbf{M} + b\mathbf{K}$$

Caughey generalization:

$$\mathbf{D} = \mathbf{M}\sum_{k=0}^{p-1} \alpha_k (\mathbf{M}^{-1}\mathbf{K})^k$$

Viscoelastic material - effect of velocity/strain-rate:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

where $\dot{\varepsilon}$ is the strain-rate.

4. Mass matrices and lumping techniques

Consistent mass matrix $\mathbf{M}_C = \int_\Omega \varrho \mathbf{N}^T \mathbf{N} \, \mathrm{d}\Omega$

Diagonal mass matrix $\mathbf{M}_L$

Averaged mass matrix $\mathbf{M}_A = \beta \mathbf{M}_C + (1 - \beta) \mathbf{M}_L$

Higher-order mass matrix

Mass lumping (diagonalization)

Row sum method:
$$m_{ii}^e = \sum_{j=1}^{n} m_{ij}^e$$

The row sum method produces negative diagonal terms for higher-order FEM.

The HRZ (Hinton-Rock-Zienkiewicz) method[1]:
A scaling method for conserving of total element mass. Procedure is as follows.

1. For each coordinate direction, select the DOFs that contribute to motion in that direction. From this set, separate translational DOF and rotational DOF subsets.
2. Add up the CMM diagonal entries pertaining to the translational DOF subset only. Call the sum S.
3. Apportion M_e to DLMM entries of both subsets on dividing the CMM diagonal entries by S.
4. Repeat for all coordinate directions.

The HRZ method can be used for higher-order FEM or FEM with rotation DOFs (beams, plates, shells).

[1] Hinton, E., Rock, T.A. & Zienkiewicz, O.C.: A note on mass lumping and related processes in the finite element method. Int. J. Earthquake Eng. Struct. Dyn., **4**, pp 245–249, 1976.

Direct inversion of mass matrix of consistent type

In explicit time integration, we need to solve

$$\ddot{\mathbf{u}} = \mathbf{M}^{\text{-1}} \left(\mathbf{f}^{ext} - \mathbf{f}^{int} - \mathbf{f}^{contact}\right)$$

The aim is to take the direct inversion of the mass matrix $\mathbf{M}^{\text{-1}}$ from the consistent mass $\mathbf{M}$ satisfies following properties

- To find a simple algorithm for a direct assembling of inversion of mass matrix in the finite element method.
- The reciprocal (inverse) mass matrix should be of suitable sparsity as the consistent mass matrix.
- It should accurately keep both low and intermediate-frequency response components;
- Except for discontinuous wave propagation problems, its numerically stable explicit integration step size should be much larger than employing the standard mass matrix.
- Its inverse should be inexpensive to generate, preferably without factorization computations.
- The customization techniques or mass scaling and tailoring should be applied, application for controlling of the maximum eigen-frequency with respect to stability in explicit time integration.

Direct inversion of mass matrix of consistent type

Reciprocal mass matrix - approximation of inversion of mass matrix $\mathbf{M}$[2], [3]

$$\mathbf{M}^{-1} = \mathbf{A}^{-\mathsf{T}}\mathbf{C}\mathbf{A}^{-1} \tag{2}$$

where $\mathbf{C}$ has the same sparsity as the global mass matrix $\mathbf{M}$ and $\mathbf{A}$ is a diagonal matrix.
Kinetic energy via velocity field $\dot{\boldsymbol{u}}$

$$T = \int_\Omega \frac{1}{2}\rho\dot{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{u}}(\boldsymbol{x},t)\,d\Omega \approx \frac{1}{2}\dot{\mathbf{u}}(t)^{\mathsf{T}}\mathbf{M}\dot{\mathbf{u}}(t) \tag{3}$$

Kinetic energy via linear momentum field ($\boldsymbol{p} = \rho\dot{\boldsymbol{u}}$)

$$T = \int_\Omega \frac{1}{2\rho}\boldsymbol{p}(\boldsymbol{x},t)\cdot\boldsymbol{p}(\boldsymbol{x},t)\,d\Omega \approx \frac{1}{2}\mathbf{p}(t)^{\mathsf{T}}\mathbf{C}\mathbf{p}(t) \tag{4}$$

[2] J. Gonzalez, R. Kolman, S.S. Cho, C. Felippa, K.C. Park. (2018) Inverse Mass Matrix via the Method of Localized Lagrange Multipliers. *International Journal for Numerical Methods in Engineering*, pp. 277–295, Vol. 113(2).

[3] J. Gonzalez, J. Kopačka, R. Kolman, S.S. Cho, K.C. Park. (2019) Inverse Mass Matrix for Isogeometric Explicit Transient Analysis via the Method of Localized Lagrange Multipliers. *International Journal for Numerical Methods in Engineering*, pp. 939–966., Vol. 117(9).

Direct inversion of mass matrix of consistent type

Potential applications:

- Direct time integration in structural dynamics and contact-impact problems.
- Time stepping based on momentum-energy approaches.
- Estimation of time step size for explicit time integration.
- Evaluation of damping matrix in structural dynamics.
- Modal identification and Component mode synthesis.
- Preconditioning for eigen-value problems.
- Others?

Direct inversion of mass matrix of consistent type

Algorithm for the reciprocal mass matrix[4], [5]

For $e = 1 ... N_e$ (elements)
Compute parametrized element mass matrix:
$\mathbf{M}_e = (1-\beta)\mathbf{M}_e^C + \beta \mathbf{M}_e^L$
Assemble diagonal projection matrix:
$\mathbf{A}_e = \mathbf{M}_e^L \quad \longrightarrow \quad$ assembly $\mathbf{A}$
Assemble reciprocal mass matrix:
$\mathbf{C}_e = \mathbf{A}_e^\mathsf{T} \mathbf{M}_e^{-1} \mathbf{A}_e \quad \longrightarrow \quad$ assembly $\mathbf{C}$
End for
Compute the free-floating inverse mass matrix: $\mathbf{M}^{-1} = \mathbf{A}^{-\mathsf{T}} \mathbf{C} \mathbf{A}^{-1}$
Obtain the projector: $\mathbf{P} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{B}\left[\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\mathbf{B}\right]^{-1}\mathbf{B}^\mathsf{T}$
Apply boundary conditions by projection: $\mathbf{M}_b^{-1} = \mathbf{P}\mathbf{M}^{-1}$
Eliminate the rows and columns of $\mathbf{M}_b^{-1}$ with applied boundary conditions

[4] J. Gonzalez, R. Kolman, S.S. Cho, C. Felippa, K.C. Park. (2018) Inverse Mass Matrix via the Method of Localized Lagrange Multipliers. *International Journal for Numerical Methods in Engineering*, pp. 277–295, Vol. 113(2).

[5] J. Gonzalez, J. Kopačka, R. Kolman, S.S. Cho, K.C. Park. (2019) Inverse Mass Matrix for Isogeometric Explicit Transient Analysis via the Method of Localized Lagrange Multipliers. *International Journal for Numerical Methods in Engineering*, pp. 939–966., Vol. 117(9).

5. Explicit time integration in FEM

nodal displacement vector: $\mathbf{u}(t)$
nodal velocity vector : $\dot{\mathbf{u}}(t) = \mathbf{v}(t)$
nodal acceleration vector : $\ddot{\mathbf{u}}(t) = \mathbf{a}(t)$

Solutions of discretized equations of motion

- modal superposition (linear problems)
- matrix exponential (linear problems)
- direct time integration (linear and nonlinear problems)

System of second order ordinary differential equations:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}^{ext}(t) - \mathbf{f}^{contact}(t) \tag{5}$$

In direct time integration, approximation of quantities at discrete time t^n

$\mathbf{u}(t^n) \approx \mathbf{u}^h(t^n) = \mathbf{u}^n$

Temporal discretization:

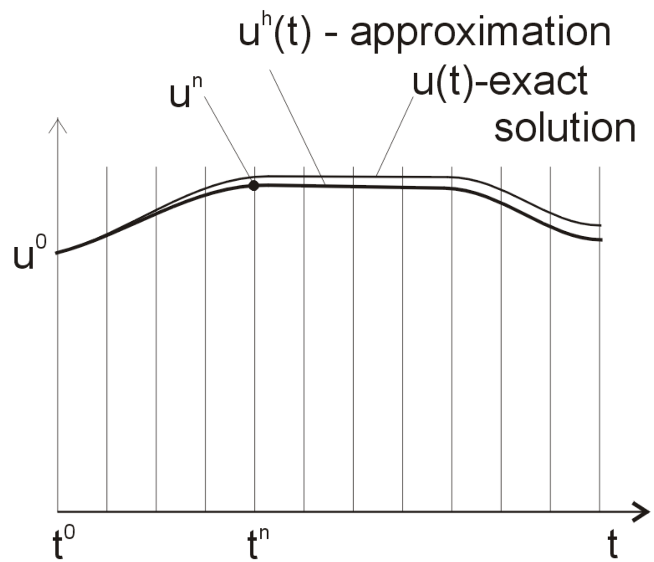
$t = 0, t^1, t^2, t^3, \ldots, T$

Time step size:

$\Delta t^i = t^{i+1} - t^i$

For constant time step size Δt:

$t^n = n\Delta t, n = 0, 1, 2, \ldots, N$



Solutions of discretized equations of motion

Mathematical methods for numerical solution of

the first-order system

$\dot{\mathbf{y}} = f(\mathbf{y}, t), \mathbf{y} = (\mathbf{u}, \dot{\mathbf{u}})^{\mathsf{T}}$ – state space

- The forward Euler method
- The backward Euler method
- The generalized trapeziodal method
- The midpoint method
- Methods of the Runge-Kutta type
- The central difference method
- Linear multi-step methods
- Other methods

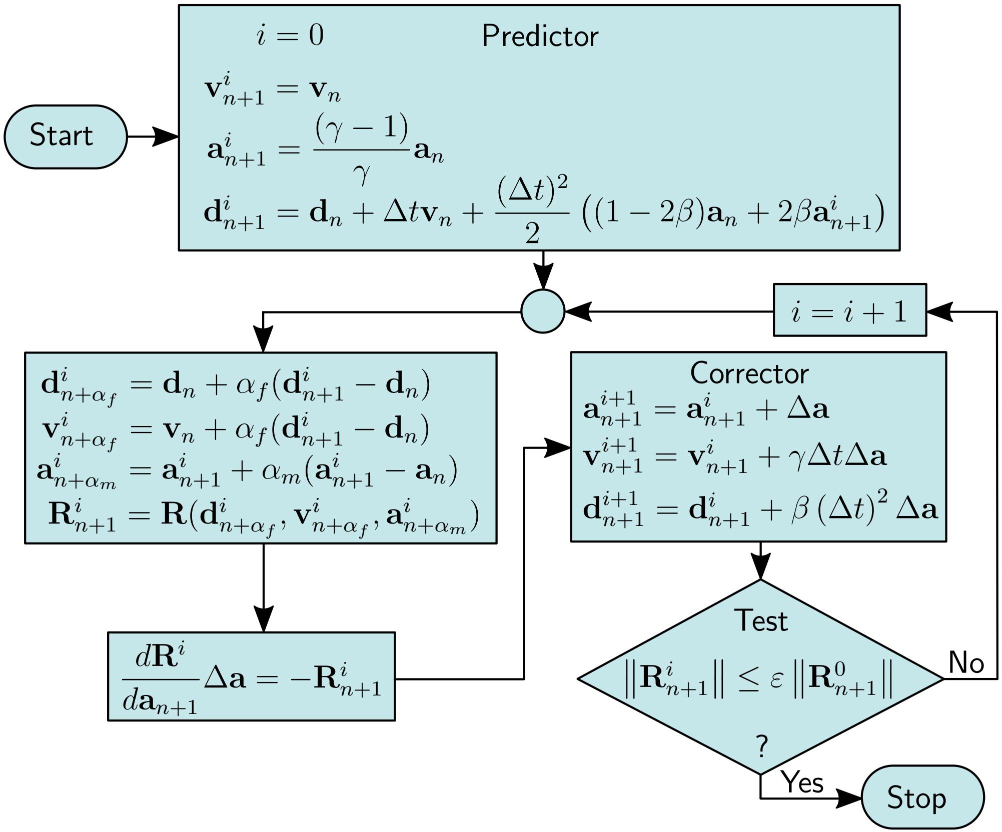
the second-order system

$\ddot{\mathbf{u}} = f(\mathbf{u}, \dot{\mathbf{u}}, t)$

- The Newmark method
- The Houbolt method
- The Wilson θ method
- The Midpoint method
- The Central difference method
- The HHT method
- The Generalized-α method
- Other methods

A predictor/multi-corrector form of time scheme

The generalized-α method [Chung, Hulbert 1993]



A review of explicit time integration methods in FEM

- **the central difference method** [Krieg 1973, Dokainish & Subbaraj 1989]
- **the Verlet method** [Verlet 1967] (molecular dynamics)
- the Trujillo method [Trujillo 1977]
- the Park variable-step central difference method [K.C. Park & Underwood 1980]
- the Chung and Lee method [Chung & Lee 1994]
- **the explicit form of the generalized-α method** [Hulbert & Chung 1996]
- the Zhai method [Zhai 1996]
- **the Tchamwa–Wielgosz method** [Tchamwa & Conway & Wielgosz 1999]
- **the explicit predictor/multi-corrector method** [Hughes 2000]
- the Tamma $et\ al.$ method [Tamma $et\ al.$ 2003]
- the Chang pseudo-dynamic method [Chang 2008]
- the semi-explicit modified mass method [Doyen $et\ al.$ 2011]
- the Yin method [Yin 2013]
- the two-time step Bathe method [Noh & Bathe 2013]
- **the multi-time step Park method** [Park et al. 2012, Cho et al. 2013, Kolman et al. 2016]

Numerical errors, properties of time integrators

Numerical errors:

- dispersion (distortion of pulse), anisotropy and diffraction, polarization errors
- **spurious oscillations**, parasitic modes
- numerical dissipation and attenuation
- period elongation and amplification

Requirements and properties of explicit methods:

- diagonal mass and damping matrices
- second-order accuracy
- symplectic and energy and momentum conserving
- unconditionally/conditionally stability, time step size estimator
- numerical dissipation controlled by a parameter
- the numerical dissipation should affect higher modes; lower modes should not be affected
- **an effective evaluator of RHS, underintegration of linear FEs with Hourglass controlling**.

6. Central difference method in time

Leapfrog integration - numerical analysis

Verlet method - molecular dynamic simulation

The central difference method

Equations of motion at the time t:

$$\mathbf{M}\ddot{\mathbf{u}}^t = \mathbf{f}^{ext} - \mathbf{f}^{int} - \mathbf{f}^{cont}$$

Approximation of time derivatives - Central difference scheme[6] in time:

$$\dot{\mathbf{u}}^t \approx \frac{\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t-\Delta t}}{2\Delta t} \qquad \ddot{\mathbf{u}}^t \approx \frac{\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^t + \mathbf{u}^{t-\Delta t}}{\Delta t^2}$$

[6] Dokainish M.A., Subbaraj K. A survey of direct time-integration methods in computational structural dynamics - I. Explicit methods. *Comput. & Struct.*, **32**(6), 1371–1386, 1989.

The central difference method

The Newmark method with $\beta = 0$**,** $\gamma = 1/2$.

Kinematic quantities:

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \Delta t\,\dot{\mathbf{u}}^t + \frac{\Delta t^2}{2}\ddot{\mathbf{u}}^t$$

$$\dot{\mathbf{u}}^{t+\Delta t} = \dot{\mathbf{u}}^t + \frac{\Delta t}{2}\left(\ddot{\mathbf{u}}^t + \ddot{\mathbf{u}}^{t+\Delta t}\right)$$

Equations of motion at the time t:

$$\mathbf{M}\ddot{\mathbf{u}}^t + \mathbf{K}\mathbf{u}^t = \mathbf{f}^t_{ext}$$

Approximation of velocity and acceleration by the **central differencies**:

$$\dot{\mathbf{u}}^t \approx \frac{1}{2\Delta t}\left(\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t-\Delta t}\right)$$

$$\ddot{\mathbf{u}}^t \approx \frac{1}{\Delta t^2}\left(\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^t + \mathbf{u}^{t-\Delta t}\right)$$

Implementation I

$$\mathbf{F}^t_{\text{eff}} = \mathbf{F}^t_{\text{ext}} - \left[\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right]\mathbf{u}^t - \frac{1}{\Delta t^2}\mathbf{M}\mathbf{u}^{t-\Delta t}$$

$$\mathbf{M}_{\text{eff}} = \frac{1}{\Delta t^2}\mathbf{M}$$

$$\mathbf{u}^{t+\Delta t} = \mathbf{M}^{-1}_{\text{eff}}\boldsymbol{F}^t_{\text{eff}}$$

In memory: displacements $\mathbf{u}^{t+\Delta t}$, $\mathbf{u}^t$, $\mathbf{u}^{t-\Delta t}$

The rest of quantities are computed if they are needed.

Implementation II - Leapfrog method

Solve time $t = 0$:
Evaluate force residual: $\mathbf{r}^0 = \mathbf{f}_{ext}(t = 0) - \mathbf{K}\mathbf{u}^0$
Compute acceleration: $\ddot{\mathbf{u}}^0 = \mathbf{M}^{-1}\mathbf{r}^0$
for $n = 1...N$ (time steps)
Evaluate force residual: $\mathbf{r}^n = \mathbf{f}^n_{ext} - \mathbf{K}\mathbf{u}^n$
Compute nodal accelerations: $\ddot{\mathbf{u}}^n = \mathbf{M}^{-1}\mathbf{r}^n$
Update nodal velocities: $\dot{\mathbf{u}}^{n+1/2} = \dot{\mathbf{u}}^{n-1/2} + \Delta t\ddot{\mathbf{u}}^n$
Update nodal displacements: $\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\dot{\mathbf{u}}^{n+1/2}$
end for

In memory: displacements $\mathbf{u}^{t+\Delta t}$, velocities $\dot{\mathbf{u}}^{t+\Delta/2}$, accelerations $\ddot{\mathbf{u}}^t$

Implementation III - Predictor-corrector form, Verlet method

Predictor

$$\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}^n$$

$$\dot{\tilde{\mathbf{u}}}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2} \ddot{\mathbf{u}}^n$$

$$\ddot{\tilde{\mathbf{u}}}^{n+1} = \mathbf{0}$$

Solve equations of motion at the time $t^{n+1} = t^n + \Delta t$

$$\mathbf{M} \Delta \ddot{\tilde{\mathbf{u}}}^{n+1} = \mathbf{f}_{ext}(t^{n+1}) - \mathbf{f}_{int}(t^{n+1}, \tilde{\mathbf{u}}^{n+1}, \dot{\tilde{\mathbf{u}}}^{n+1}) - \mathbf{f}_{cont}(t^{n+1}, \tilde{\mathbf{u}}^{n+1}, \dot{\tilde{\mathbf{u}}}^{n+1})$$

Corrector

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1}$$

$$\dot{\mathbf{u}}^{n+1} = \dot{\tilde{\mathbf{u}}}^{n+1} + \frac{\Delta t}{2} \Delta \ddot{\tilde{\mathbf{u}}}^{n+1}$$

$$\ddot{\mathbf{u}}^{n+1} = \Delta \ddot{\tilde{\mathbf{u}}}^{n+1}$$

Advantage: in memory only $\mathbf{u}^{t+\Delta t}$, $\mathbf{v}^{t+\Delta t}$, $\mathbf{a}^{t+\Delta t}$

Central difference method

Requirement:

- for efficient computations, it is needed the inversion of $\mathbf{M}$
- lumped (diagonal) mass matrix - no required a linear solver

Properties:

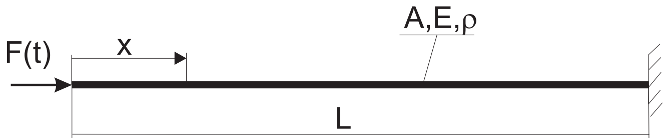
- explicit method
- conditionally stable (time step can not be chosen arbitrary)
- second order accuracy
- conserving of total energy in the limit $\Delta t \rightarrow 0$, energy oscillations in sense of the shadow Hamiltonian
- no amplitude decay
- period shortening
- reversible in time

One-dimensional stress wave in a bar

Linear (classical) wave equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}$$

u - displacement, x - position, t - time, $c_0 = \sqrt{E/\rho}$ - wave speed



Scheme of a free-fixed bar under an impact loading.

Loading

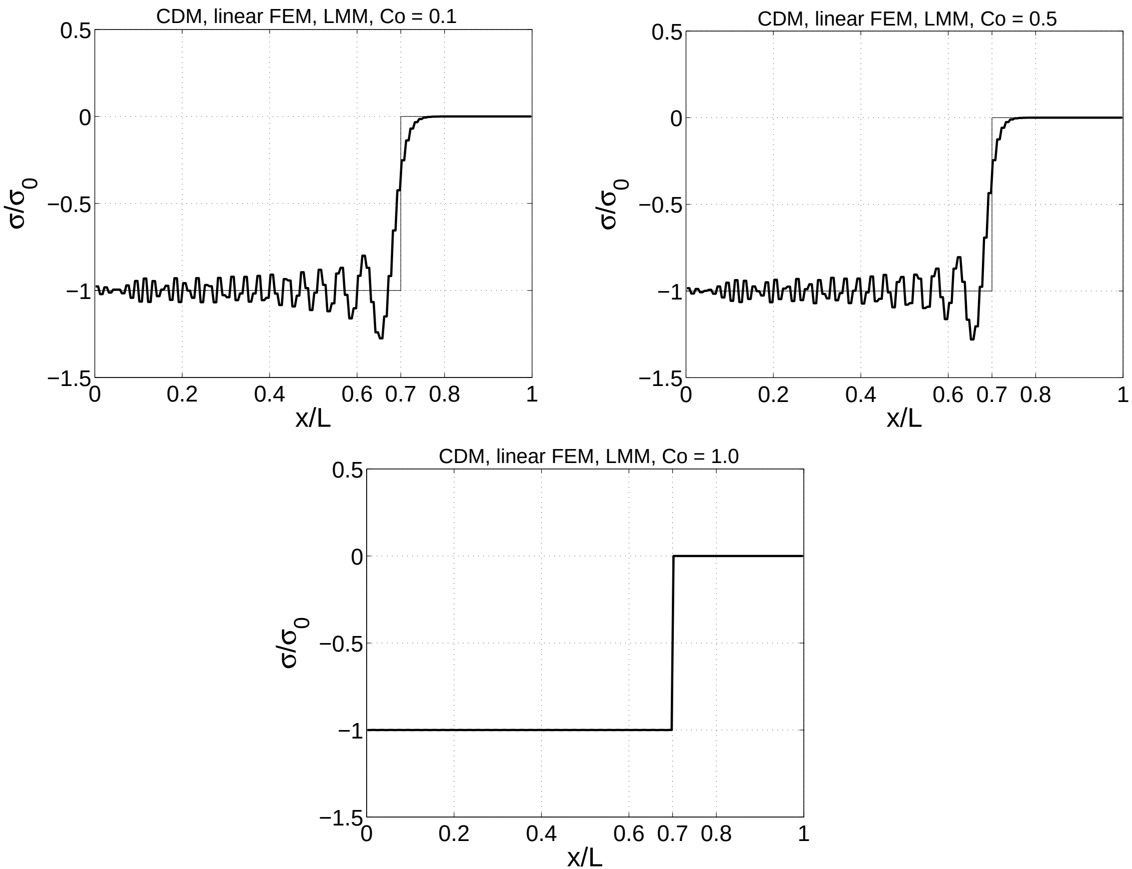
$$\sigma(0, t) = -\sigma_0 H(t)$$

σ is the stress, H is the Heaviside step function.
Analytical solution

$$\sigma(x, t) = -\sigma_0 H(c_0 t - x)$$

KF Graff. Wave motion in elastic solids. Oxford University Press, 1975

One dimensional wave test



7. Solving of nonlinear time-depend problems

Vector of internal forces:

$$\mathbf{f}_{int} = \int_\Omega \mathbf{B}^T \boldsymbol{\sigma}(\text{deformation tensor, strain-rate, time, temperature, internal variables})\, \mathrm{d}\Omega$$

Often, vector local internal forces are evaluated by one-point Gauss integration (only one integration point taken place in the centroid of a finite element) and the stress tensor $\boldsymbol{\sigma}$ are kept as an internal state variable.

One-point integration $\Rightarrow$ Hourglass stabilization
Enhanced FEM

Solving of nonlinear time-depend problems

Algorithm:
Initial conditions and initialization, time $t = 0$.
Set initial displacement $\mathbf{u}^0$, initial velocity $\dot{\mathbf{u}}^0$, initial stress $\boldsymbol{\sigma}^0$ and initial values of other internal material variables
$n = 0$, compute $\mathbf{M}$ or $\mathbf{M}^{-1}$
Evaluate internal force $\mathbf{f}^n_{int}$, evaluate external force $\mathbf{f}^n_{ext}$, evaluate contact force $\mathbf{f}^n_{cont}$
Evaluate force residual: $\boldsymbol{r}^n = \boldsymbol{f}^n_{ext} - \boldsymbol{f}^n_{int} - \boldsymbol{f}^n_{cont}$
Compute accelerations $\ddot{\boldsymbol{u}}^n = \boldsymbol{M}^{-1}\boldsymbol{r}^n$
Time update: $t^{n+1} = t^n + \Delta t^{n+1/2}$, $t^{n+1/2} = \frac{1}{2}(t^n + t^{n+1})$

- Update nodal velocities $\dot{\boldsymbol{u}}^{n+1/2} = \dot{\boldsymbol{u}}^n + (t^{n+1/2} - t^n)\ddot{\boldsymbol{u}}^n$
- Enforce velocity boundary conditions
- Update nodal displacements $\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t^{n+1/2}\dot{\boldsymbol{u}}^{n+1/2}$
- Evaluate internal force $\mathbf{f}^{n+1}_{int}$ for $\boldsymbol{u}^{n+1}$ - the most demanding operations
- Compute force residual $\boldsymbol{r}^{n+1}$ at t^{n+1} and accelerations $\ddot{\boldsymbol{u}}^{n+1}$
- Update nodal velocities $\dot{\boldsymbol{u}}^{n+1} = \dot{\boldsymbol{u}}^{n+1/2} + (t^{n+1} - t^{n+1/2})\ddot{\boldsymbol{u}}^{n+1}$
- Check energy balance at the time step $n + 1$
- Update counter $n = n + 1$

Goto to STEP TIME UPDATE

8. Dynamic relaxation

General nonlinear problems in residual form:

$$\mathbf{r} = \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{f}^{int}(\mathbf{u}, \boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) - \mathbf{f}^{ext}(t) = \mathbf{0} \tag{6}$$

Time t is a parametr.

Cardinal question is setting the diagonal damping matrix $\mathbf{D}$ so that velocity $\dot{\mathbf{u}}(t) \rightarrow \mathbf{0}$ and acceleration $\ddot{\mathbf{u}}(t) \rightarrow \mathbf{0}$ in explicit time integration. After that, $\mathbf{f}^{int}(\mathbf{u}, \boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) - \mathbf{f}^{ext}(t) \rightarrow \mathbf{0}$.
Optimal setting of damping parameters with respect to convergence, computational cost and damping of dominant eigen-frequencies.

9. Stability of time schemes

Stability theory

In direct time integration, the recursive relationship in time stepping process has a form

$$\begin{bmatrix} \mathbf{u}^{t+\Delta t} \\ \dot{\mathbf{u}}^{t+\Delta t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{u}^{t} \\ \dot{\mathbf{u}}^{t} \end{bmatrix} + \mathbf{L}^{t+\nu}(r), \tag{7}$$

where $\mathbf{A}$ marks the amplification operator, which dictates stability behaviour of the method.

We define the spectral radius of $\mathbf{A}$

$$\rho\left(\mathbf{A}\right) = \max_{i=1,2,\dots,n} \left|\lambda_i\right|, \tag{8}$$

where λ_i denotes the i-the eigen value of the operator $\mathbf{A}$

Stability criterion yields:

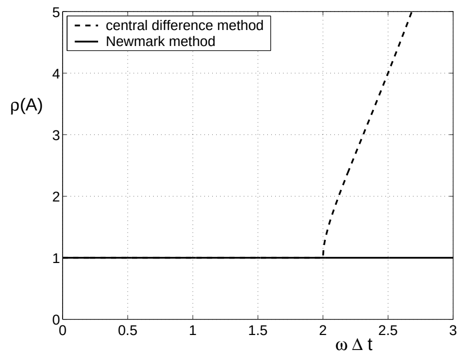
1. if all eigenvalues are distinct, it must be satisfied $\rho\left(\mathbf{A}\right) \leq 1$ whereas
2. If $\mathbf{A}$ contains multiple eigenvalues, we require that all such eigenvalues $|\lambda_i| < 1$.

Stability theory - Central difference method

$$\mathbf{A} = \begin{bmatrix} 2 - \omega^2 \Delta t^2, & -1 \\ 1, & 0 \end{bmatrix}, \tag{9}$$

with eigen-values

$$\lambda_{1,2} = \frac{2 - \omega^2 \Delta t^2}{2} \pm \sqrt{\frac{(2 - \omega^2 \Delta t^2)^2}{4} - 1}. \tag{10}$$



The central difference method is **conditionally stable**.
Stability limit for the central difference method

$$\Delta t \omega \leq 2 \tag{11}$$

It yields the stability formula for the time step size Δt as

$$\Delta t \leq \frac{2}{\omega_{max}} \tag{12}$$

where ω_{max} is the maximum eigen value of the discretized system.

10. Time step size estimations for FEM

Time step size estimations for FEM

We define the Courant number:

$$Co = \frac{\Delta t c_1}{H}$$

Δt - time step size, c_1 is wave speed of longitudinal wave, H - characteristic length (length of finite element edge)

The non-dimensional angular velocity:

$$\bar{\omega} = \frac{\omega H}{c_1}$$

The critical time step size for the cetral difference method

$$\Delta t_{cr} = \frac{2}{\omega_{max}}$$

Then, the critical time step size is given

$$Co_{cr} = \frac{\Delta t_{cr} c_1}{H} = \frac{2}{\bar{\omega}_{max}}$$

Time step size estimations for FEM

Stability limit for the central difference method $\Delta t \leq \Delta t_{cr} = \frac{2}{\omega_{max}}$

Methods of time step size estimations

- **global methods** (computation or estimation of ω_{max}, $\mathbf{K\Phi} = \omega^2 \mathbf{M\Phi}$)

  - ω_{max} can be computed or estimated using global mass and stiffness matrices.

- **element based methods** (computation or estimation of ω^e_{max} on elemental level,
  $\mathbf{K}^e \mathbf{\Phi^e} = \omega_e^2 \mathbf{M}^e \mathbf{\Phi}_e$

  - respecting the element eigenvalue inequality $\omega_{\max} \leq \max_i \omega_i^e$ over all finite elements.

  The highest eigenvalue of dissassembled system is higher than the highest eigenvalue of the assembled system.

- **nodal based methods** - ω_{max} can be estimated from nodal stiffness and mass properties based on the Gershgorin's theorem

Time step size estimations for FEM

Global methods

Power iteration:

$$\lambda \mathbf{\Phi}_{n+1} = \mathbf{A}\mathbf{\Phi_n} \qquad \mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$$

Algorithm:

1. Initialize eigenvector $\mathbf{\Phi}_0$, e.g. randon in range [-1,1], $i = 0$
2. i=i+1
3. Compute $\mathbf{\Psi}_{i+1} = \mathbf{K}\mathbf{\Phi}_i$ or as internal force $\mathbf{\Psi}_{i+1} = \mathbf{f}_{int}(\mathbf{\Phi}_i)$
4. Compute $\chi_{i+1} = \mathbf{M}^{-1}\mathbf{\Psi}_{i+1}$
5. Compute estimate of eigenvalue $\lambda_{i+1}^{max} = \|\chi_{i+1}\|$
6. Update eigenvector $\mathbf{\Phi}_{i+1} = \chi_{i+1}/\lambda_{i+1}^{max}$
7. If $|\lambda_{i+1}^{max}/\lambda_i^{max} - 1| > \epsilon$ or $i < N^{iter}$ go to STEP 2.

About 10 iterations are sufficient.

Time step size estimations for FEM

Element based methods

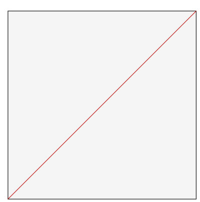
Upper bound for eigenfrequency

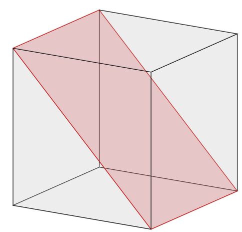
$$\omega_{\max} \leq \max_i \omega_i^e \leq \frac{c_1}{l_e}$$

with longitudinal wave speed c and characteristic length of element l_e.

How to choose l_e?[7]

$$l_e^{2D} = \frac{A_{element}}{l_{\max}} \qquad l_e^{3D} = \frac{V_{element}}{A_{\max}}$$





CFL (Courant-Friedrichs-Lewy [8]) condition $\Delta t \leq \alpha \frac{l_e}{c_1}$, α depends on element type, integration type, order, shape, mass matrix, mass scaling, etc.

[7] LS-DYNA manual

[8] Courant, R., Friedrichs, K., Lewy, H., 1967, On the partial difference equations of mathematical physics

Critical Courant number

Linear 1D FEM with the lumped mass matrix

$$\omega^h_{max} = \frac{2c_0}{H}, \qquad \bar{\omega}^h_{max} = 2, \qquad Co_{cr} = \frac{\Delta t_{cr}\, c_0}{H} = \frac{2}{\bar{\omega}^h_{max}} = 1$$

Linear 1D FEM with the consistent mass matrix

$$\omega^h_{max} = \frac{\sqrt{12}c_0}{H}, \qquad \bar{\omega} = \sqrt{12}, \qquad Co_{cr} = \frac{\Delta t_{cr}\, c_0}{H} = \frac{2}{\bar{\omega}^h_{max}} = 1/\sqrt{3} \approx 0.577$$

Square linear 2D and 3D FEM the with diagonal mass matrix

$$Co_{crit} = \frac{\Delta t_{crit}\, c_1}{H} = 1$$

Serendipity quadratic (eight-noded) 2D and 3D FEM with the lumped mass by the HRZ method

$$Co_{crit} = \frac{\Delta t_{crit}\, c_1}{H} \approx 0.2$$

Time step size estimations for FEM

Nodal based methods

Gershgorin circle theorem[9] based method: For a given square matrix $\boldsymbol{A}$ (complex $n \times n$ matrix) the Gershgorin's circle which belongs to the i-th diagonal entry A_{ii} is defined as $S_i(A_{ii}, \mathcal{R}_i = \sum_{j=1, i \neq j}^{n} \mid A_{ij} \mid), \; i = 1, ..., n$

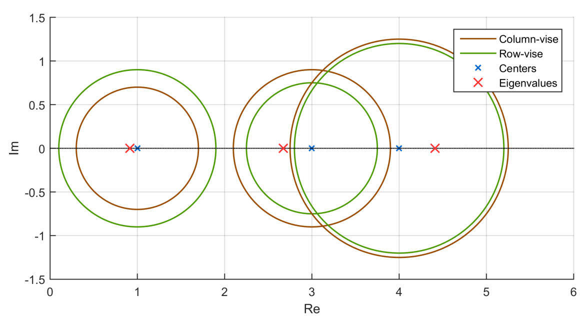
where S_i defines a circle with radius $\mathcal{R}_i$ and position around x-axis at the position A_{ii}.

Gershgorin's circle: $\quad D(A_{ii}, R_i) \qquad R_i = \sum_{j=1, i \neq j}^{n} \mid A_{ij} \mid$

Every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$.

Example: $\boldsymbol{A} = \begin{bmatrix} 3 & -0.5 & 0.4 \\ -0.75 & 4 & -0.5 \\ 0 & -0.7 & 1 \end{bmatrix}$

row-wise	**column-wise**
$D(3, 0.9)$	$D(3, 0.75)$
$D(4, 1.25)$	$D(4, 1.2)$
$D(1, 0.7)$	$D(1, 0.9)$



[9] Gerschgorin, S., 1931, Über die Abgrenzung der Eigenwerte einer Matrix

Time step size estimations for FEM

Nodal based methods

Application for FEM with the lumped mass matrix[10]:

$$\omega_{\max}^2 \ \leq \ \max_i \frac{\sum_{j=1}^{n} \mid K_{ij} \mid}{M_{ii}}$$

This method respects Dirichlet boundary conditions.

Application for FEM with lumped mass matrix in contact-impact problems using penalty formulation

$$\omega_{\max}^2 \ \leq \ \max_i \frac{\sum_{j=1}^{n} \mid K_{ij} \mid + K_i^p}{M_{ii}}$$

where $\mathbf{K}^p$ is the corresponding penalized stiffness matrix.

[10] Kulak, R., F., 1989, Critical Time Step Estimation for Three-Dimensional Explicit Impact Analysis

11. Mass scaling

Motivation: change frequency spectrum of FEM model via modification of mass matrix, affect maximum eigen-frequency of FE system so that the critical time step is larger and computations is efficient.

Smaller maximum eigen-value $\Rightarrow$ larger time step size

Modification of mass matrix as

$$\mathbf{M}^o = \mathbf{M} + \mathbf{\Lambda}^o$$

where $\mathbf{\Lambda}^o$ is the artificial added mass matrix.

Mass scaling

Methods of mass scaling in FEM

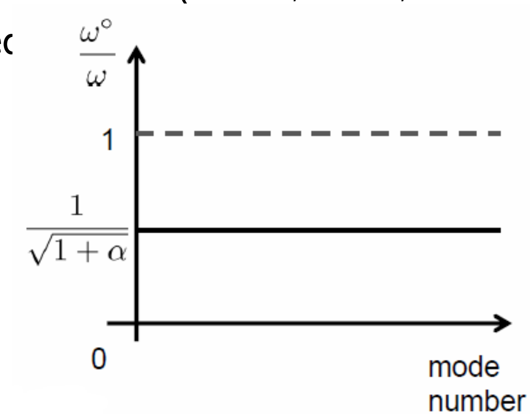
- **convential mass scaling** - adding artificial mass in diagonal terms of mass matrix

$$\mathbf{m}_e^{\lambda} = \frac{\rho A l_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{m}_e^{o} = \mathbf{m}_e + \alpha \mathbf{m}_e^{\lambda}$$

- preserving the diagonal structure of mass matrix
- increasing element inertia - applied only to a small number of element - applied to structural finite element (beam, shell, solid-like shell, applied only on rotation degrees of freec

Frequency spectrum:



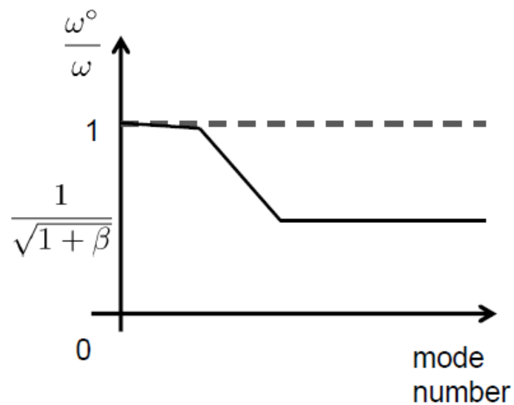
- **selective mass matrix**[11] - adding artificial mass so so that translation inertia is preserving.

$$\mathbf{m}_e^\lambda = \beta \frac{\rho A l_e}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{m}_e^o = \mathbf{m}_e + \beta \mathbf{m}_e^\lambda$$

- only selected modes are affected
- off-diagonal mass matrix structure ⇒ using the reciprocal mass matrix

Frequency spectrum:



[11] Olovsson Etal. (2005) Selective Mass Scaling for explicit Finite Element Analyses, IJNME 63

General form for preserving of translation inertia[12]

$$\mathbf{m}_e^o = \frac{\Delta m}{n-1}(\mathbf{I} - \sum_{i=1}^{n} \mathbf{o}_i \mathbf{o}_i^{\mathsf{T}})$$

For example for 2D, rigid body modes for a four-noded element are chosen as

$$\mathbf{o}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^{\mathsf{T}}$$

$$\mathbf{o}_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^{\mathsf{T}}$$

[12] Olovsson, et al. (2005) Selective Mass Scaling for explicit Finite Element Analyses, IJNME

General form for elimination of selected eigen-modes with corresponding modal vectors $\mathbf{\Phi}_l$ [13]

$$\mathbf{m}_e^o = \alpha \mathbf{P}_e \mathbf{m}_e \mathbf{P}_e^\intercal$$

where

$$\mathbf{P}_e = \mathbf{I} - \mathbf{\Phi}_l [\mathbf{\Phi}_l^\intercal \mathbf{\Phi}_l]^{-1} \mathbf{\Phi}_l$$

Modifying of the mass matrix so that the total mass is preserved and the higher frequency spectrum is improved.

Mass scaled matrix:

$$\mathbf{m}_e^o = \mathbf{m}_e + \mathbf{m}_e^\lambda$$

with

$$\mathbf{m}_e^\lambda = \mathbf{M}_e \mathbf{\Phi}_e^h \mathbf{S} \mathbf{\Phi}_e^{h\intercal} \mathbf{M}_e^\intercal \qquad (13)$$

where $\mathbf{\Phi}^h$ contents the higher mode shapes corresponding to eigen-modes for improving, $\mathbf{S}$ is the diagonal matrix with coefficients for cutting of value of higher eigen-frequencies[14].

[13] J. Gonzalez, et al. (2018) Inverse Mass Matrix via the Method of Localized Lagrange Multipliers IJNME.
[14] J. Gonzalez, K.C. Park (2019) Largestep explicit time integration via mass matrix tailoring, IJNME.

Numerical tests - Eigen-vibration problems

1D bi-material rod - linear FEM

$L_1 = 5\,\mathrm{m},\ L_2 = 5\,\mathrm{m},$

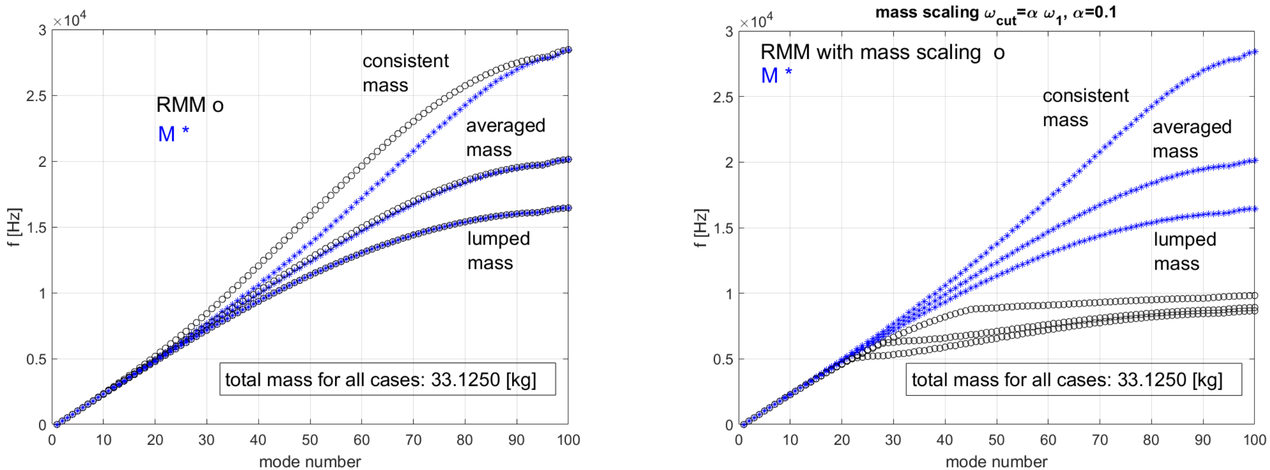
$A_1 = 10 \cdot 10^{-4}\,\mathrm{m}^2,\ A_2 = 5 \cdot 10^{-4}\,\mathrm{m}^2,$

$\rho_1 = 2700\,\mathrm{kg} \cdot \mathrm{m}^{-3},\ \rho_2 = 7850\,\mathrm{kg} \cdot \mathrm{m}^{-3},$

$E_1 = 69e9\,\mathrm{Pa},\ E_2 = 210e9\,\mathrm{Pa},$

number of elements $nel_1 = 50,\ nel_2 = 50$

total mass $m = A_1 L_1 \rho_1 + A_2 L_2 \rho_2 = 33.1250\,\mathrm{kg}$



(a) frequency spectrum

(b) frequency spectrum with mass tailoring

12. Application of Dirichlet boundary conditions in explicit time schemes

- Direct elimination
- Penalty/bipenalty method
- Lagrange multipliers

Application of Dirichlet boundary conditions via Lagrange multipliers

Constrained elastodynamic problem

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{B}\boldsymbol{\lambda} + \mathbf{K}\mathbf{u} = \mathbf{f}_{ext} \tag{14}$$

$$\mathbf{B}^{\intercal}\ddot{\mathbf{u}} - \mathbf{L}_b\ddot{\mathbf{u}}_b = \mathbf{0} \tag{15}$$

A close formula for the Lagrange multipliers (reaction forces on constraints)

$$\boldsymbol{\lambda} = (\mathbf{B}^{\intercal}\mathbf{M}^{-1}\mathbf{B})^{-1}(\mathbf{B}^{\intercal}\mathbf{M}^{-1}\mathbf{r} - \mathbf{L}_b\ddot{\mathbf{u}}_b) \tag{16}$$

with the residual $\mathbf{r} = \mathbf{f}_{ext} - \mathbf{K}\mathbf{u}$

Application of Dirichlet boundary conditions via Lagrange multipliers

Initialize $t^0 = 0$, $\mathbf{u}^0$ and $\dot{\mathbf{u}}^0$ respecting the constraints $\mathbf{B}\mathbf{u}^0 = \mathbf{0}$ and $\mathbf{B}\dot{\mathbf{u}}^0 = \mathbf{0}$, assemble $\mathbf{M}^{-1}$, $\mathbf{K}$, $\mathbf{B}$ and compute $\ddot{\mathbf{u}}^0 = \mathbf{M}^{-1}\left(\mathbf{f}^0_{ext} - \mathbf{K}\mathbf{u}^0\right)$, $\mathbf{u}^{\frac{1}{2}} = \mathbf{u}^0 + \Delta t\dot{\mathbf{u}}^0$

While t $<$ T

Setting of the time step size Δt by the power iteration method

$\mathbf{u}^n = \mathbf{u}^{n-1} + \Delta t\dot{\mathbf{u}}^{n-\frac{1}{2}}$

$\mathbf{r}^n = \mathbf{f}^n_{ext} - \mathbf{K}\mathbf{u}^n$

$\boldsymbol{\lambda}^n = (\mathbf{B}^{\mathsf{T}}\mathbf{M}^{-1}\mathbf{B})^{-1}(\mathbf{B}^{\mathsf{T}}\mathbf{M}^{-1}\mathbf{r}^n - \mathbf{L}_b\ddot{\mathbf{u}}^n_b)$

$\ddot{\mathbf{u}}^n = \mathbf{M}^{-1}\left(\mathbf{r}^n - \mathbf{B}\lambda^n\right)$

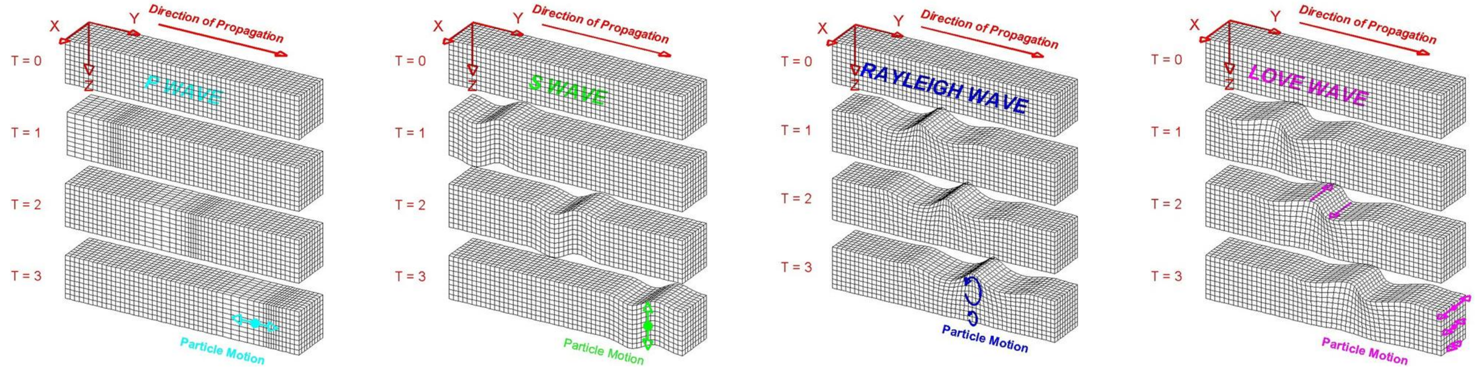
$\dot{\mathbf{u}}^{n+\frac{1}{2}} = \dot{\mathbf{u}}^{n-\frac{1}{2}} + \Delta t\ddot{\mathbf{u}}^n$

$t = t + \Delta t$; $n = n + 1$;

13. Wave speeds in solids, dispersion of FEM, mesh size and time step size for explicit FEM

Waves in isotropic elastic continuum

P-wave S-wave Rayleigh's wave Love's wave



source:©2007 Michigan Technological University; http://www.geo.mtu.edu/UPSeis/waves.html

Other wave types:

- waves in rods, flexural (bending) and torsional waves, guided waves
- Lamb's waves (waves in plates, dispersive, application in NDT)
- surface Rayleigh's waves (waves in a half-space)
- Love's waves (waves in a half-space covered by a layer with different elastic properties)
- von Schmidt's waves (reflected waves from boundaries)
- inter-facial Stoneley's (Leaky Rayleigh's) waves
- Scholte's waves (solid-liquid interface)

Wave speeds in solids

3D longitudinal wave:	$c_1 = \sqrt{(\Lambda + 2G)/\rho}$
3D shear wave:	$c_2 = \sqrt{G/\rho}$
2D longitudinal wave under plane strain state:	$c_1 = \sqrt{(\Lambda + 2G)/\rho}$
2D shear wave under plane strain state:	$c_2 = \sqrt{G/\rho}$
2D longitudinal wave under plane stress state:	$c_1 = \sqrt{\dfrac{E}{(1-\nu^2)\,\rho}}$
2D shear wave under plane stress state:	$c_2 = \sqrt{G/\rho}$
1D longitudinal wave under uniaxial strain state:	$c = \sqrt{\dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)\,\rho}}$
1D longitudinal wave under uniaxial stress state:	$c = \sqrt{E/\rho}$

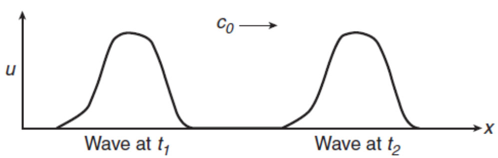
Λ, G Lamé are parameters, ν is Poisson ratio.

Dispersion

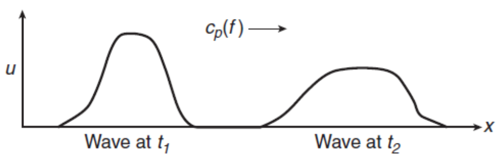
Dispersion: dependence of angular velocity ω on wave number k

Nonliner Dispersion law: $\omega = f(k) \Rightarrow$ **distortion of pulse**

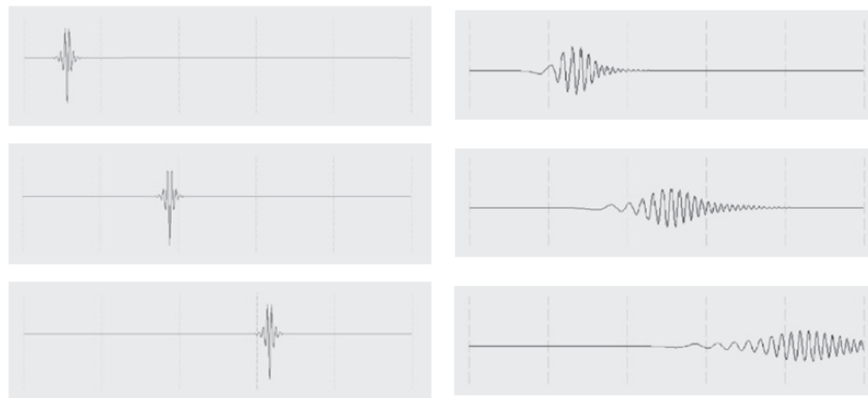
Group speed: $c_g = \frac{\partial \omega}{\partial k}$; Phase speed: $c = \frac{\omega}{k}$



(a) Example of wave traveling without distortion



(b) Example of wave envelope traveling with distortion


(a) Non-dispersive

(b) Dispersive

Type of dispersion:

- physical dispersion (higher order terms in governing equations)
- geometrical dispersion (wave in plates or cylinders)
- numerical dispersion (FEM, etc.)

Spatial dispersion - one-atomic chain

Brillouin, L.: *Wave Propagation in Periodic Structures.*
Dover Publications, Inc., New York 1953.

Equation of motion :

$\ddot{u}_j = \omega_0^2 (u_{j-1} - 2u_j + u_{j+1})$

$\omega_0^2 = c_0^2 = K/m$

Assumption of solution:

$u_j(t) = U_0 e^{\mathrm{i}j\psi} e^{\mathrm{i}\omega t}$

where $\psi = k + \mathrm{i}b$, $k \in < -\pi, \pi >$

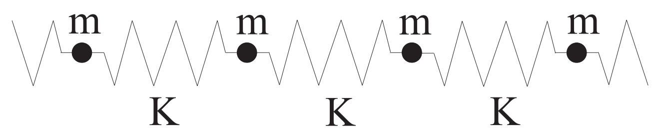
Propagating wave for $\omega/\omega_0 < 2$

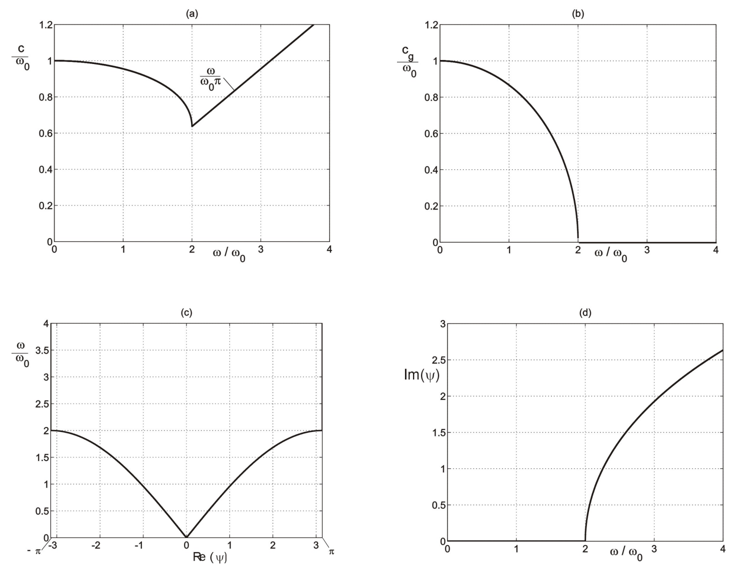
$k \neq 0$ a $b = 0$

dispersion relation $\omega = 2\omega_0 \left|\sin(k/2)\right|$

Attenuating wave for $\omega/\omega_0 > 2$

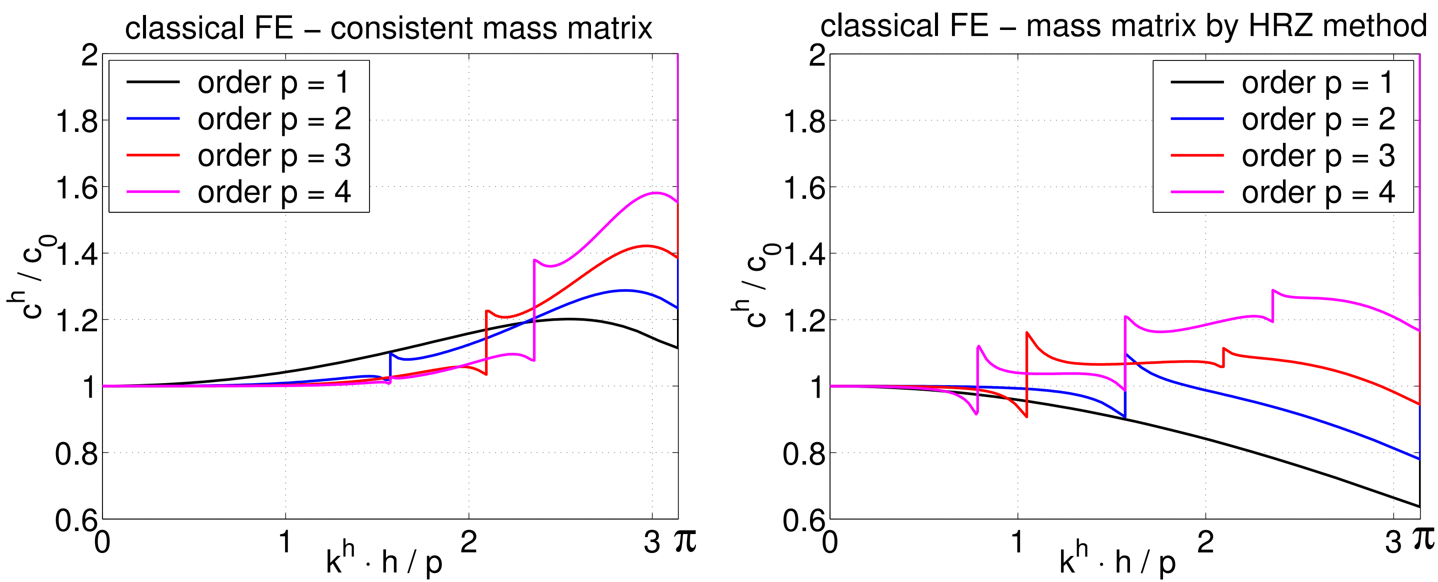
$k = \pi$ and $b \neq 0$





Dispersion - classical 1D FEM

Thompson, L.L., Pinsky, P.M.: *Complex wavenumber Fourier analysis of the p-version finite element method.* *Computational Mechanics*, Vol. 13(4), 255-275, 1994.
Kolman, R., Plešek J., Okrouhlik, M. Complex wavenumber Fourier analysis of the B-spline based finite element method. *Wave Motion*, Vol. 51(2), 348359,2014.



The high mode behaviour of Lagrangian FE is divergent with the order of approximation. Existing of optical modes for classical FEs.

Dispersion - classical 2D FEM [a]

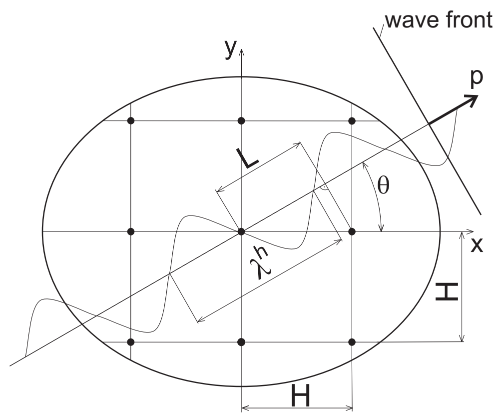
[a] Kolman R. Plešek J., Okrouhlik M., Gabriel D. Grid dispersion analysis of plane square biquadratic serendipity finite elements in transient elastodynamics, International Journal for Numerical Methods in Engineering, **96**(1), pp. 1–28, 2013.

Characteristic equations of motion for patch

$$\mathbf{M}_c \ddot{\mathbf{u}}^h + \mathbf{K}_c \mathbf{u}^h = \mathbf{0}$$

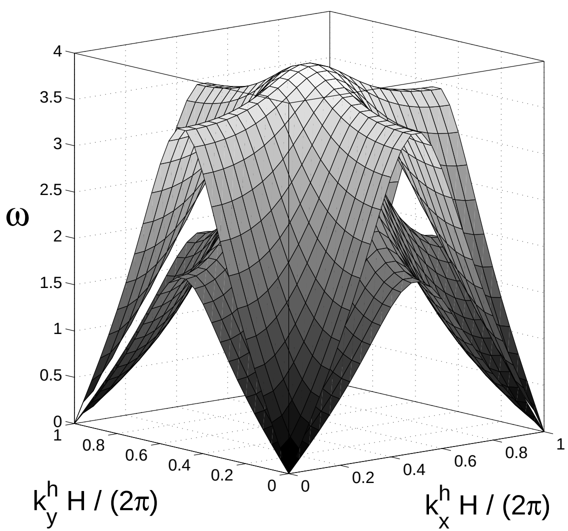
Fourier analysis - prescriebed time nodal displacements

$$u^h_{mn} = U_{mn} \exp\left[\mathrm{i}\left(k^h x_m p_x + k^h y_n p_y - \omega t\right)\right]$$
$$v^h_{mn} = V_{mn} \exp\left[\mathrm{i}\left(k^h x_m p_x + k^h y_n p_y - \omega t\right)\right]$$

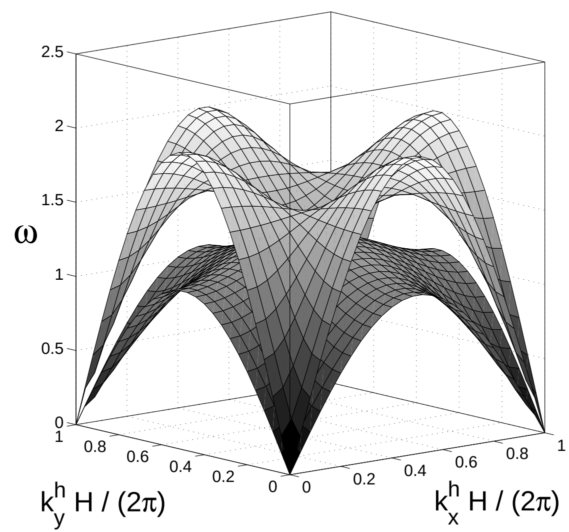


Bilinear FEM - dispersion relationship

Consistent mass matrix



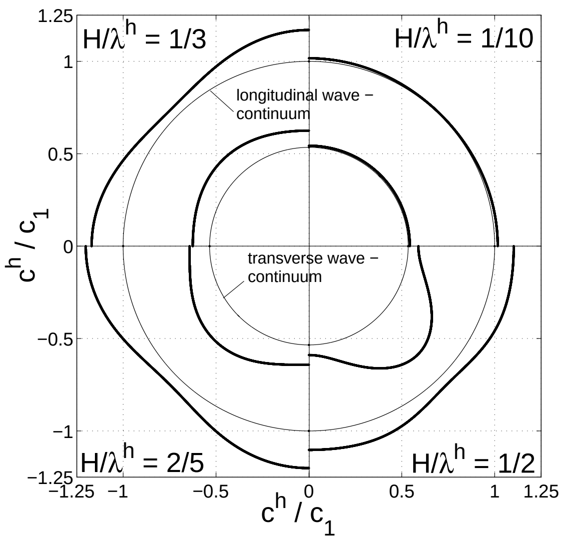
Lumped mass matrix



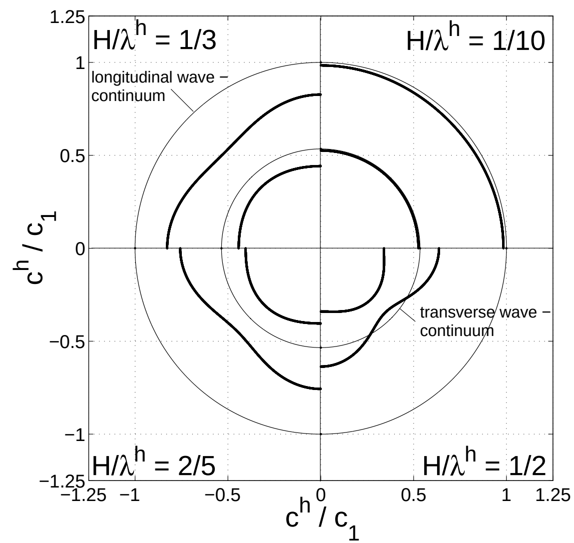
Two solutions: longitudinal wave and shear wave

Bilinear FEM - polar diagrams

Consistent mass matrix



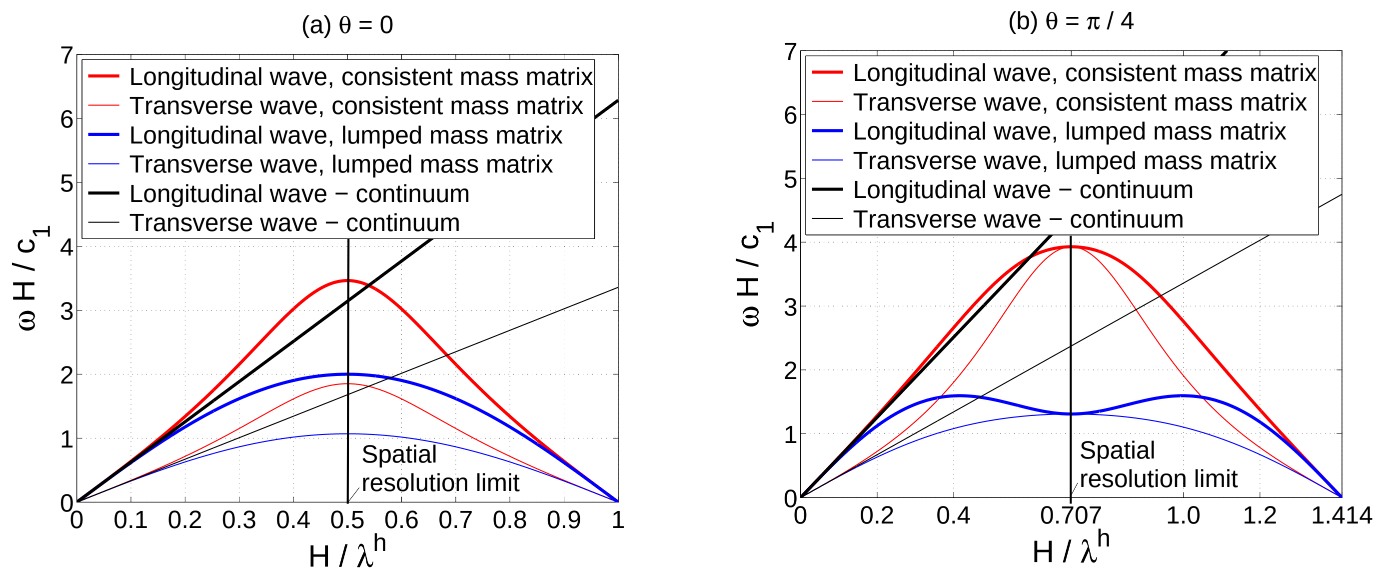
Lumped mass matrix



Anisotropic effect of FE discretization.

Bilinear finite element

Dispersion curves



Consistent mass matrix ⇒ **overestimation** of wave speed ⇒ the Newmark method

Lumped mass matrix ⇒ **underestimation** of wave speed ⇒ the central difference method

Mesh size recommendation

Mesh size:
$$H \le (H/\lambda^h)_{allowed}\, \lambda,$$

$$H \le (H/\lambda^h)_{allowed}\, \frac{c_2}{f_{max}},$$

where f_{max} is the highest loading frequency.

$(H/\lambda^h)_{allowed}$				
speed	c^h	c^h_g	c^h	c^h_g
error [%]	linear		serendipity	
1	0.080	0.043	0.325	0.215
2	0.110	0.059	0.394	0.259
5	0.162	0.090	-	0.333
10	0.225	0.132	-	0.405

For quadratic element and for dispersion error in phase speed 2% is recommended mesh size as $\mathbf{H} < \lambda/\mathbf{3}$.

For linear element and for dispersion error in phase speed 2% is recommended mesh size as $\mathbf{H} < \lambda/\mathbf{10}$.

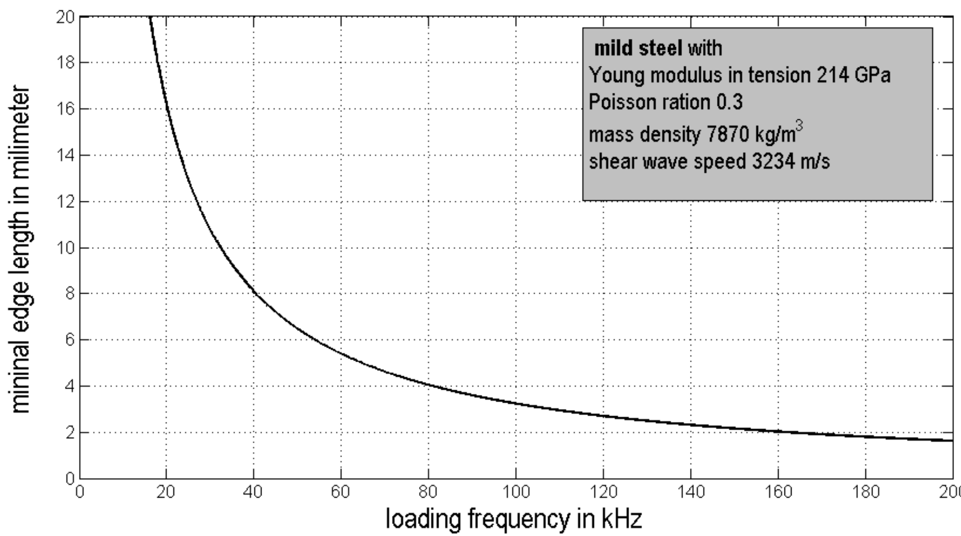
Example of mesh size estimation for bilinear FEM

Minimal edge length for structured bilinear FE meshes:

$$H_{min} \leq 10\lambda^h_{min},$$

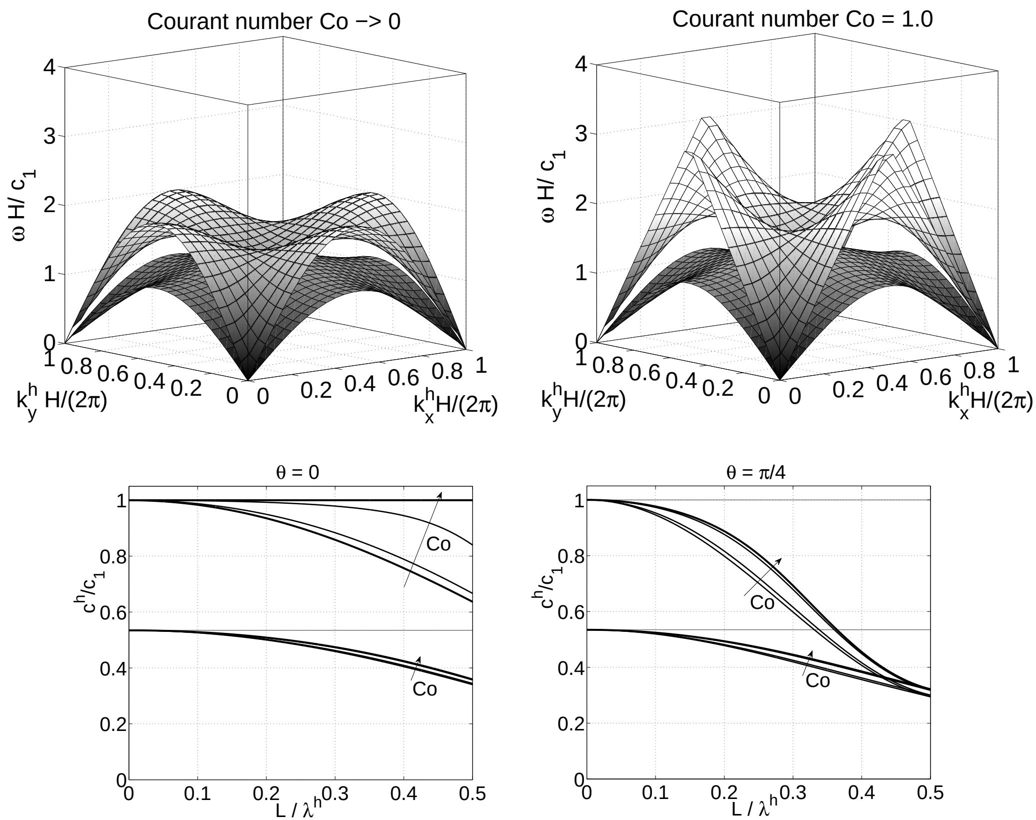
where the minimal wave length propagating wave with frequency f_{max} can be estimated as

$$\lambda^h_{min} \approx \frac{c_2}{f_{max}} = \frac{\text{shear wave speed}}{\text{loading frequency}}.$$



Temporal-spatial dispersion - linear FEM with diagonal mass matrix and CD[a]

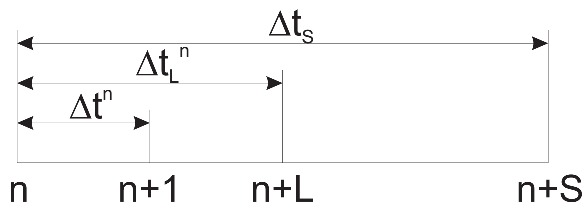
[a] Kolman R. Plešek J., Červ J. ,Okrouhlik M., Pařík P. Temporal-spatial dispersion and stability analysis of finite element method in explicit elastodynamics, International Journal for Numerical Methods in Engineering, **106**(2), pp. 113–128, 2016.



Reducing numerical dispersion in wave propagation

Integration of longitudinal and shear waves separately [15,16]

The mismatch in wave speeds of shear, longitudinal and other of wave types $\Rightarrow$ dispersion errors.



Longitudinal waves (under plane strain): $\Delta t_L = H/c_L, \quad c_L = \sqrt{\frac{\Lambda+2G}{\rho}}$

Transverse waves (under plane strain): $\Delta t_S = H/c_S, \quad c_S = \sqrt{\frac{G}{\rho}}$

$$c_S < c_L \Rightarrow \Delta t_L < \Delta t_S$$

Stability limit: $\Delta t_c = \Delta t_L$. Time step: $\Delta t = \alpha_L \Delta t_c$.

[15] K.C. Park, S.J. Lim, H. Huh. (2012) A method for computation of discontinuous wave propagation in heterogeneous solids: Basic algorithm description and application to one-dimensional problems. IJNME

[16] S.S. Cho, K.C. Park, H. Huh. A method for multidimensional wave propagation analysis via component-wise partition of longitudinal and shear waves (2013) INJME. 2013.

Proposed scheme - 1D case

Park K.C., Lim S.J., Huh H. A method for computation of discontinuous wave propagation in heterogeneous solids: basic algorithm description and application to one–dimensional problems. *Inter. J. Num. Meth. Eng.*, **91**(6), 622–643, 2012.

Pushforward extrapolation

$$\mathbf{u}^{n+c} = \mathbf{u}^n + \Delta t_c \dot{\mathbf{u}}^n + \frac{(\Delta t_c)^2}{2} \ddot{\mathbf{u}}^n$$

$$\ddot{\mathbf{u}}^{n+c} = \mathbf{M}^{-1}\Big(\mathbf{f}^{ext}(t^{n+c}) - \mathbf{f}^{int}(\mathbf{u}^{n+c})\Big), \;\; t^{n+c} = t^n + \Delta t_c$$

Pullback interpolation

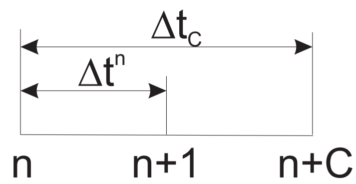
$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + {\Delta t_c}^2 \beta_1(\alpha) \ddot{\mathbf{u}}^n + {\Delta t_c}^2 \beta_2(\alpha) \ddot{\mathbf{u}}^{n+c}$$

$$\beta_1(\alpha) = \frac{1}{6}\alpha\Big(1 + 3\alpha - \alpha^2\Big), \quad \beta_2(\alpha) = \frac{1}{6}\alpha\Big(\alpha^2 - 1\Big), \quad \alpha = \frac{\Delta t}{\Delta t_c}$$

As alluded to already, this scheme filters out post-shock oscillations but triggers front-shock oscillations. The best choice of α is 0.5.
For $\alpha = 1 \rightarrow$ the central difference method.

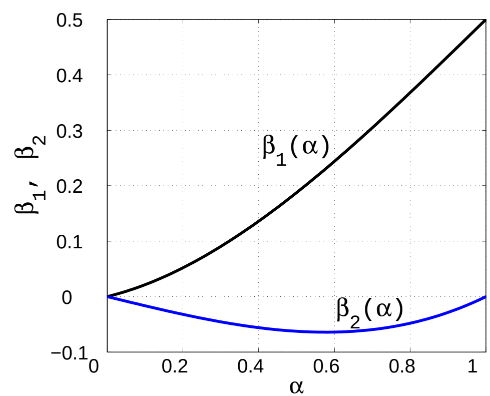
The pullback interpolation

Pullback interpolation: $\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + {\Delta t_c}^2 \beta_1(\alpha) \ddot{\mathbf{u}}^n + {\Delta t_c}^2 \beta_2(\alpha) \ddot{\mathbf{u}}^{n+c}$



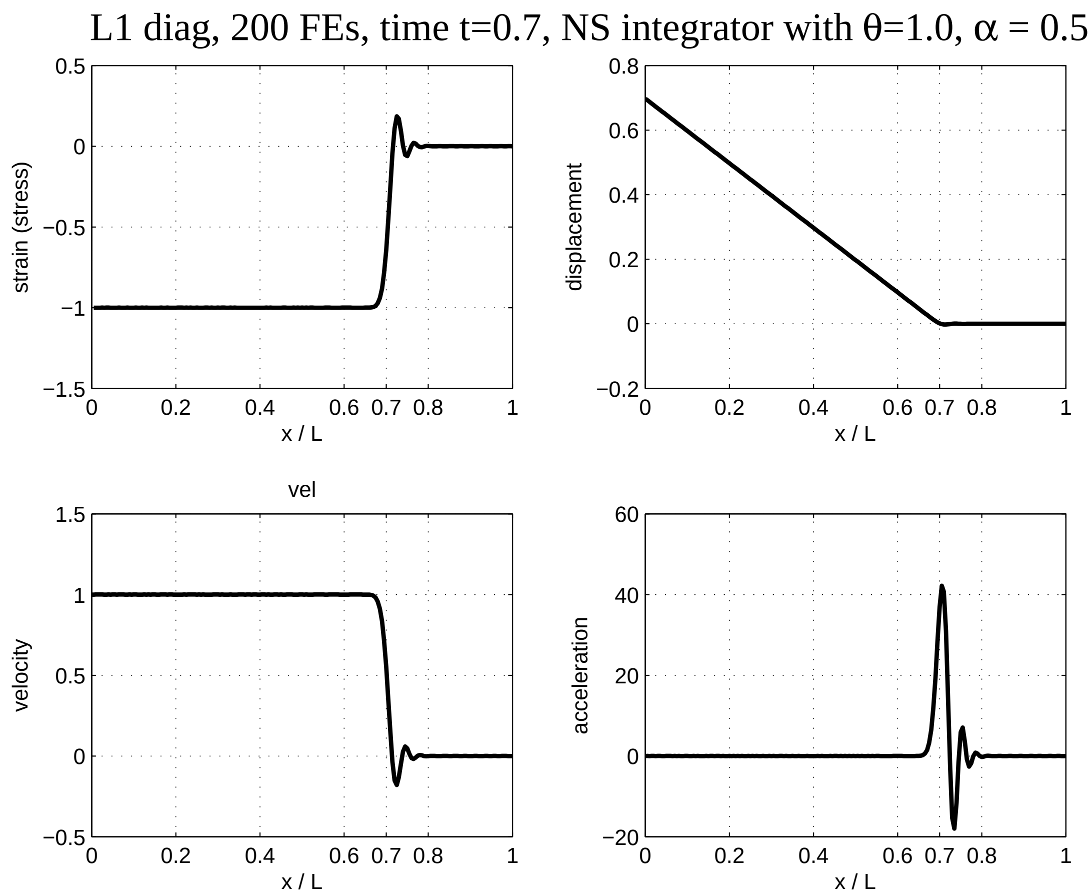
$$\beta_1(\alpha) = \frac{1}{6}\alpha\left(1 + 3\alpha - \alpha^2\right)$$

$$\beta_2(\alpha) = \frac{1}{6}\alpha\left(\alpha^2 - 1\right)$$



Stability limit with respect to the CFL condition $\Delta t_c = H/c_0, \quad c_0 = \sqrt{E/\rho}$.
Time step size: $\Delta t \leq \Delta t_c$, $\Delta t = \alpha \Delta t_c$, $\alpha = (0,1]$, our choice $\alpha = 0.5$.

Results for the front-shock including integration



1D non-spurious oscillations scheme

STEP 1. A post-shock triggering integrator - the central difference method

$$\mathbf{u}_{cd}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}^n$$

$$\ddot{\mathbf{u}}_{cd}^{n+1} = \mathbf{M}^{-1}(\mathbf{f}^{ext}(t^{n+1}) - \mathbf{f}^{int}(t^{n+1}, \mathbf{u}_{cd}^{n+1})$$

$$\dot{\mathbf{u}}_{cd}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^n + \ddot{\mathbf{u}}_{cd}^{n+1})$$

STEP 2. A front-shock triggering integrator

$$\mathbf{u}^{n+c} = \mathbf{u}^n + \Delta t_c \dot{\mathbf{u}}^n + \frac{(\Delta t_c)^2}{2} \ddot{\mathbf{u}}^n$$

$$\ddot{\mathbf{u}}^{n+c} = \mathbf{M}^{-1}(\mathbf{f}^{ext}(t^{n+c}) - \mathbf{f}^{int}(t^{n+c}, \mathbf{u}^{n+c}))$$

$$\mathbf{u}_{fs}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \Delta t_c{}^2 \beta_1(\alpha) \ddot{\mathbf{u}}^n + \Delta t_c{}^2 \beta_2(\alpha) \ddot{\mathbf{u}}^{n+c}$$

$$\beta_1(\alpha) = \frac{1}{6}\alpha\left(1 + 3\alpha - \alpha^2\right), \beta_2(\alpha) = \frac{1}{6}\alpha\left(\alpha^2 - 1\right), \alpha = \frac{\Delta t}{\Delta t_c}$$

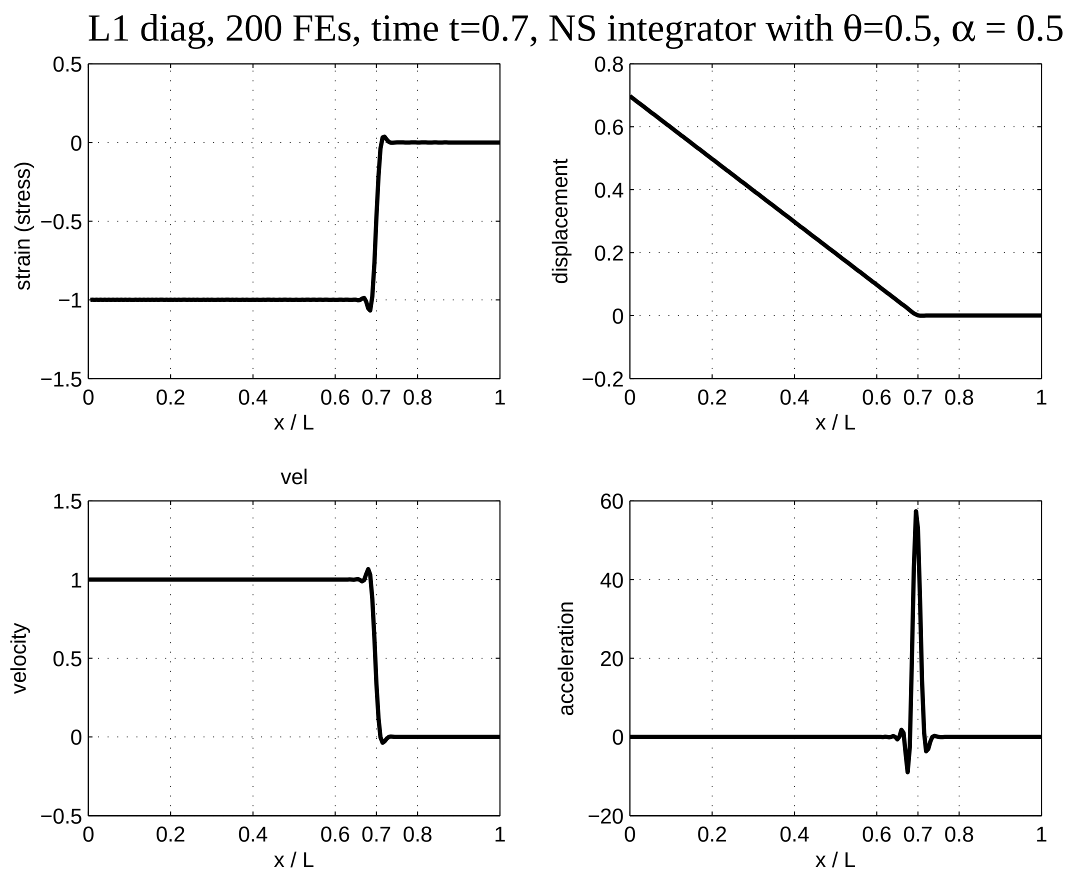
$$\ddot{\mathbf{u}}_{fs}^{n+1} = \mathbf{M}^{-1}(\mathbf{f}^{ext}(t^{n+1}) - \mathbf{f}^{int}(t^{n+1}, \mathbf{u}_{fs}^{n+1}))$$

$$\dot{\mathbf{u}}_{fs}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^n + \ddot{\mathbf{u}}_{fs}^{n+1})$$

STEP 3. Averaging STEP 1. and STEP 2. by $\theta \in [0, 1]$

$$\mathbf{u}^{n+1} = \theta \mathbf{u}_{fs}^{n+1} + (1-\theta)\mathbf{u}_{cd}^{n+1},\ \dot{\mathbf{u}}^{n+1} = \theta \dot{\mathbf{u}}_{fs}^{n+1} + (1-\theta)\dot{\mathbf{u}}_{cd}^{n+1},\ \ddot{\mathbf{u}}^{n+1} = \theta \ddot{\mathbf{u}}_{fs}^{n+1} + (1-\theta)\ddot{\mathbf{u}}_{cd}^{n+1}$$

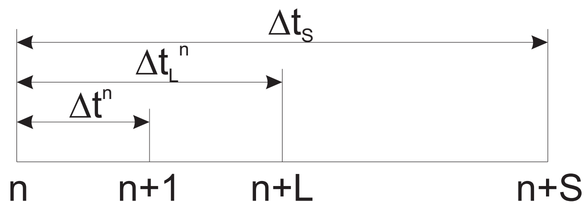
Results for the proposed time scheme



The algorithm exhibits minimal sensitivity on the time step size.

Integration of longitudinal and transverse waves together

The mismatch in wave speeds of shear, longitudinal and other of wave types $\Rightarrow$ dispersion errors.



Longitudinal waves (under plane strain)

$$\Delta t_L = H/c_L, \quad c_L = \sqrt{\frac{\Lambda + 2G}{\rho}}$$

Transverse waves (under plane strain)

$$\Delta t_S = H/c_S, \qquad c_S = \sqrt{\frac{G}{\rho}}$$

$$c_S < c_L \;\Rightarrow \Delta t_L < \Delta t_S$$

Stability limit: $\Delta t_c = \Delta t_L$. Time step: $\Delta t = \alpha_L \Delta t_c$.

Component-wise partitioned equations of motion

Decomposition of elemental displacement field : $\mathbf{u}^e = \mathbf{u}^e_L + \mathbf{u}^e_S, \quad \mathbf{u}^e_L = \mathbf{D}^e_L \mathbf{u}^e, \quad \mathbf{u}^e_S = \mathbf{D}^e_S \mathbf{u}^e$

Partition of unity: $\mathbf{D}^e_S + \mathbf{D}^e_L = \mathbf{I}^e$

Projector property: $\mathbf{D}^{e\mathsf{T}}_S \mathbf{D}^e_S = \mathbf{D}^e_S$, $\mathbf{D}^{e\mathsf{T}}_L \mathbf{D}^e_L = \mathbf{D}^e_L$

Symmetry: $\mathbf{D}^{e\mathsf{T}}_S = \mathbf{D}^e_S$, $\mathbf{D}^{e\mathsf{T}}_L = \mathbf{D}^e_L$

Orthogonality: $\mathbf{D}^e_L \mathbf{D}^e_S = \mathbf{D}^e_S \mathbf{D}^e_L = \mathbf{0}^e$

Element mass commutability: $\mathbf{D}^{e\mathsf{T}}_L \mathbf{M}^e = \mathbf{M}^e \mathbf{D}^e_L$, $\mathbf{D}^{e\mathsf{T}}_S \mathbf{M}^e = \mathbf{M}^e \mathbf{D}^e_S$

Element mass orthogonality: $\mathbf{D}^{e\mathsf{T}}_L \mathbf{M}^e \mathbf{D}^e_S = \mathbf{M}^e \mathbf{D}^e_L \mathbf{D}^e_S = \mathbf{0}^e$

The virtual work for a generic element may be written as

$$\delta\Pi^e(\mathbf{u}^e) = \delta\mathbf{u}^{e\mathsf{T}}(\mathbf{f}^e_{ext} - \mathbf{f}^e_{int} - \mathbf{M}^e\ddot{\mathbf{u}}^e)$$

The virtual work can be decomposed into the following partitioned work[17]:

$$\delta\Pi^e(\mathbf{u}^e_L, \mathbf{u}^e_S) = \underbrace{\delta\mathbf{u}^{e\mathsf{T}}_L(\mathbf{f}^e_{ext,L} - \mathbf{f}^e_{int,L} - \mathbf{M}^e\ddot{\mathbf{u}}^e_L)}_{\text{longitudinal component equation}} + \underbrace{\delta\mathbf{u}^{e\mathsf{T}}_S(\mathbf{f}^e_{ext,S} - \mathbf{f}^e_{int,S} - \mathbf{M}^e\ddot{\mathbf{u}}^e_S)}_{\text{shear component equation}}$$

where

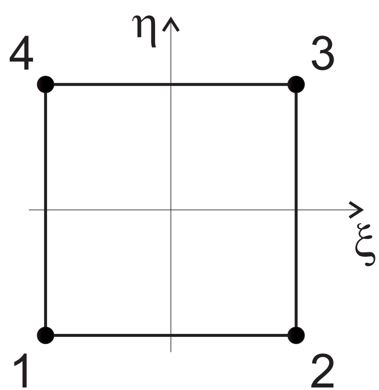
$$\mathbf{f}^e_{ext,L} = \mathbf{D}^{e\mathsf{T}}_L \mathbf{f}^e_{ext}, \quad \mathbf{f}^e_{int,L} = \mathbf{D}^{e\mathsf{T}}_L \mathbf{f}^e_{int}$$
$$\mathbf{f}^e_{ext,S} = \mathbf{D}^{e\mathsf{T}}_S \mathbf{f}^e_{ext}, \quad \mathbf{f}^e_{int,S} = \mathbf{D}^{e\mathsf{T}}_S \mathbf{f}^e_{int}$$

[17] R. Kolman, S.S. Cho, K.C. Park, Efficient implementation of an explicit partitioned shear and longitudinal wave propagation algorithm, International Journal for Numerical Methods in Engineering, 2016, vol. 107, no. 7, p. 543-579.

Decomposition of displacement and force fields

$$u(x,y,t) = \sum_{i=1}^{4} N_i(\xi,\eta) u_i, \quad v(x,y,t) = \sum_{i=1}^{4} N_i(\xi,\eta) v_i$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta),\ N_2 = \frac{1}{4}(1+\xi)(1-\eta),\ N_3 = \frac{1}{4}(1+\xi)(1+\eta),\ N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$



$$u(x,y) = u_L(x,y) + u_S(x,y)$$

$$u_L(x,y) = \tfrac{1}{4}(u_1+u_2+u_3+u_4) + \tfrac{1}{4}(-u_1+u_2+u_3-u_4)\xi + \tfrac{1}{4}(u_1-u_2+u_3-u_4)\xi\eta$$

$$u_S(x,y) = \tfrac{1}{4}(-u_1-u_2+u_3+u_4)\eta$$

$$v(x,y) = v_L(x,y) + v_S(x,y)$$

$$v_L(x,y) = \tfrac{1}{4}(v_1+v_2+v_3+v_4) + \tfrac{1}{4}(-v_1-v_2+v_3+v_4)\eta + \tfrac{1}{4}(v_1-v_2+v_3-v_4)\xi\eta$$

$$v_S(x,y) = \tfrac{1}{4}(-v_1+v_2+v_3-v_4)\xi$$

$$\mathbf{u}_L^e = \mathbf{D}_L^e \mathbf{u}^e, \quad \mathbf{u}_S^e = \mathbf{D}_S^e \mathbf{u}^e, \quad \mathbf{u}^e = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^{\mathsf{T}}$$

$$\mathbf{f}_L^e = \mathbf{D}_L^e \mathbf{f}^e, \quad \mathbf{f}_S^e = \mathbf{D}_S^e \mathbf{f}^e, \quad \mathbf{f}^e = \{f_{1x}, f_{1y}, f_{2x}, f_{2y}, f_{3x}, f_{3y}, f_{4x}, f_{4y}\}^{\mathsf{T}}$$

With one-point integration, Hourglass modes are suppressed.

Decomposed matrices for a quadrilateral

Shear decomposed matrix

$$\mathbf{D}_S^e = \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Longitudinal decomposed matrix

$$\mathbf{D}_L^e = \frac{1}{4}\begin{bmatrix} 3 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 3 \end{bmatrix}$$

A front-shock integrator in multidimen. cases

$$\mathbf{u}^{n+L} = \mathbf{u}^n + \Delta t_L \dot{\mathbf{u}}^n + \frac{(\Delta t_L)^2}{2}\ddot{\mathbf{u}}^n$$
$$\mathbf{u}^{n+S} = \mathbf{u}^n + \Delta t_L \dot{\mathbf{u}}^n + \frac{(\Delta t_S)^2}{2}\ddot{\mathbf{u}}^n$$
$$\ddot{\mathbf{u}}_L^{n+L} = \mathbf{M}^{\text{-}1}(\mathbf{f}_L^{ext}(t^{n+L}) - \mathbf{f}_L^{int}(t^{n+L}, \mathbf{u}^{n+L}))$$
$$\ddot{\mathbf{u}}_S^{n+S} = \mathbf{M}^{\text{-}1}(\mathbf{f}_S^{ext}(t^{n+S}) - \mathbf{f}_S^{int}(t^{n+L}, \mathbf{u}^{n+S}))$$

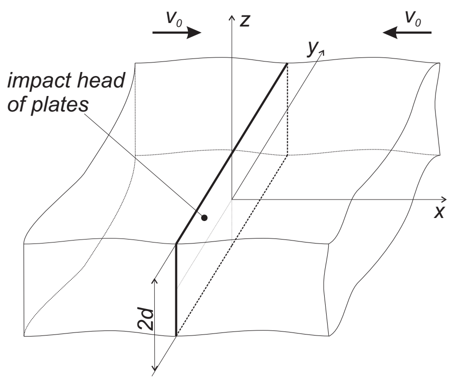
$$\mathbf{u}_L^{n+1} = \mathbf{u}_L^n + \Delta t \dot{\mathbf{u}}_L^n + \Delta {t_L}^2 \beta_1(\alpha_L)\ddot{\mathbf{u}}_L^n + \Delta {t_L}^2 \beta_2(\alpha_L)\ddot{\mathbf{u}}_L^{n+L}$$
$$\mathbf{u}_S^{n+1} = \mathbf{u}_S^n + \Delta t \dot{\mathbf{u}}_S^n + \Delta {t_S}^2 \beta_1(\alpha_S)\ddot{\mathbf{u}}_S^n + \Delta {t_S}^2 \beta_2(\alpha_S)\ddot{\mathbf{u}}_S^{n+S}$$
$$\beta_1(\alpha_{L,S}) = \frac{1}{6}\alpha_{L,S}\big(1 + 3\alpha_{L,S} - \alpha_{L,S}^2\big), \beta_2(\alpha_{L,S}) = \frac{1}{6}\alpha_{L,S}\big(\alpha_{L,S}^2 - 1\big)$$
$$\alpha_L = \frac{\Delta t}{\Delta t_L}, \alpha_S = \frac{\Delta t}{\Delta t_S}$$

$$(\mathbf{u}_{fs}^{n+1} = \mathbf{u}_L^{n+1} + \mathbf{u}_S^{n+1})$$
$$\mathbf{u}_{fs}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \Delta {t_L}^2 \beta_1(\alpha_L)\ddot{\mathbf{u}}_L^n + \Delta {t_L}^2 \beta_2(\alpha_L)\ddot{\mathbf{u}}_L^{n+L} + \Delta {t_S}^2 \beta_1(\alpha_S)\ddot{\mathbf{u}}_S^n + \Delta {t_S}^2 \beta_2(\alpha_S)\ddot{\mathbf{u}}_S^{n+S}$$
$$\ddot{\mathbf{u}}_{fs}^{n+1} = \mathbf{M}^{\text{-}1}(\mathbf{f}^{ext}(t^{n+1}) - \mathbf{f}^{int}(t^{n+1}, \mathbf{u}_{fs}^{n+1}))$$
$$\dot{\mathbf{u}}_{fs}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^n + \ddot{\mathbf{u}}_{fs}^{n+1})$$

Then, the averaging with the central difference solution at the time t^{n+1}: $\mathbf{u}_{cd}^{n+1}, \dot{\mathbf{u}}_{cd}^{n+1}, \ddot{\mathbf{u}}_{cd}^{n+1}$

14. Examples of wave propagation problems

Impact of thick elastic plates



Geometry: plate thickness $2d = 5.0$ [mm]
length $L = 4d = 10.0$ [mm]
2D problem, plane strain.
Nodes on impact head fixed.

Elastic and mass parameters:

$$E = 200 \text{ [GPa]},\ \nu = 0.3 \text{ [-]},\ \rho = 7800\,[\mathrm{kg/m^3}]$$

Initial velocity $v_0 = 1$ [m/s].

Mesh density of linear FEs: 300×300.

The proposed method with $\theta = 0.5$.

The central difference method.

Time step sizes:

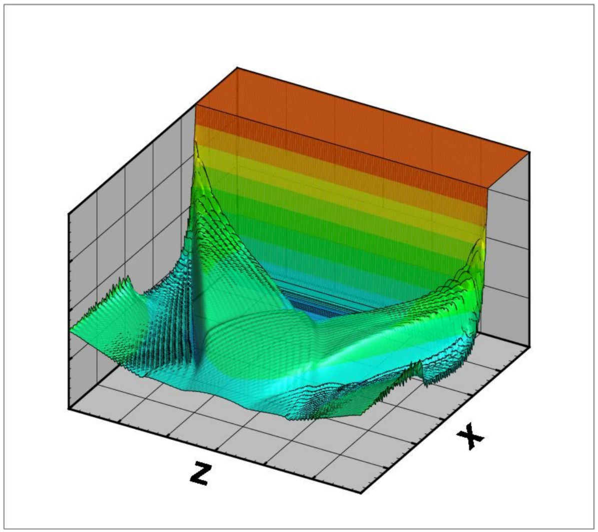
$$\Delta t = 0.5H/c_L,\ \Delta t_L = H/c_L,\ \Delta t_S = H/c_S$$

Analytical solution of the problem:

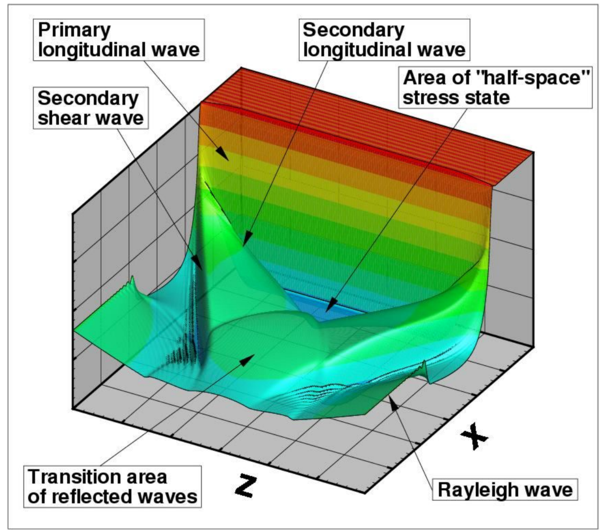
Brepta B, Valeš F. Longitudinal impact of bodies. *Acta Technica ČSAV*, **32**, 575–602, 1987.

Impact of thick elastic plates

the central difference method



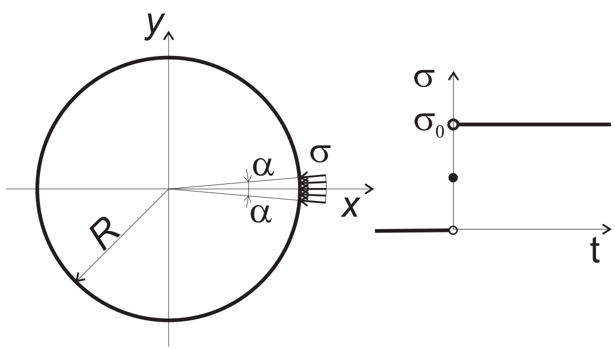
the Park's method



Distributions of σ_{xx} at the time $t = 1.5d/c_L$.

A lot of analytical/semi-analytical solutions of elastic wave propagation and impact problems have been reported at the link
http://www.cdm.cas.cz/ hora/brepta/zpravy.html

Thin elastic disc loaded by a sudden radial force



Geometry: disc radius $R = 1.0$ [mm]
2D problem, plane stress
Elastic and mass parameters:

$E = 8/9$ [GPa]

$\nu = 1/3$ [-]

$\rho = 1\,[\mathrm{kg/m^3}]$

Normal stress:

$\sigma_0 = 1$ [GPa], $\alpha = \pi/60$ [rad]

FE model - a half of disc.

Number of finite elements: 21600.

The proposed method with $\theta = 0.5$.

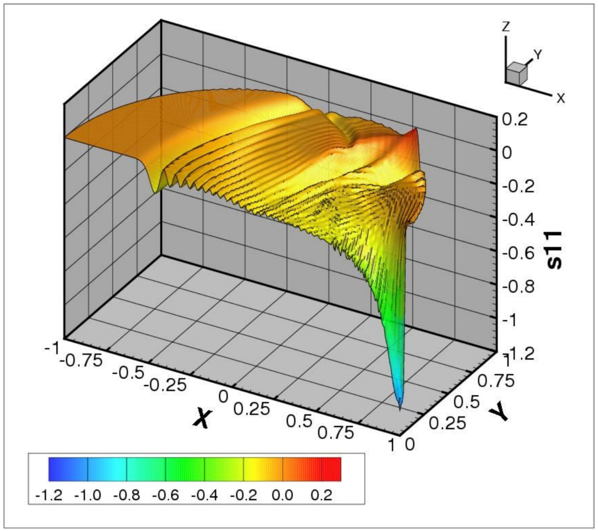
Time step sizes:
$\Delta t = 0.5\Delta t_L$, $\Delta t_L = 0.7H/c_L$
$\Delta t_S = \Delta t_L c_L/c_S$

Analytical solution of the problem:

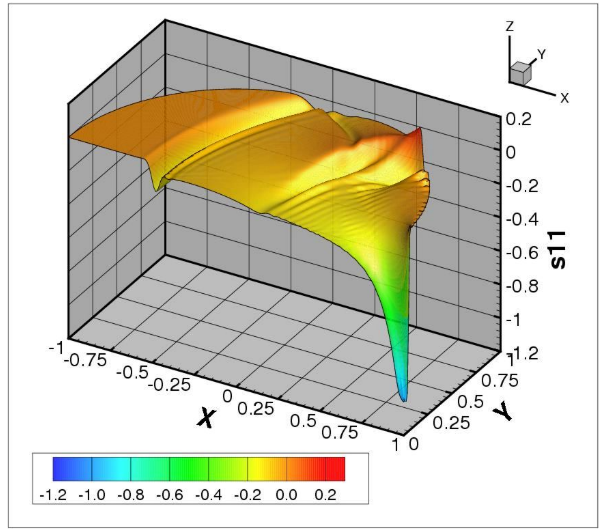
Červ J, Slavikova J. Motion and stress-state of a thin disc under radial impact load. *Acta Technica ČSAV*, **32**(2), 113–133, 1987.

Thin elastic disc loaded by a sudden radial force

the central difference method

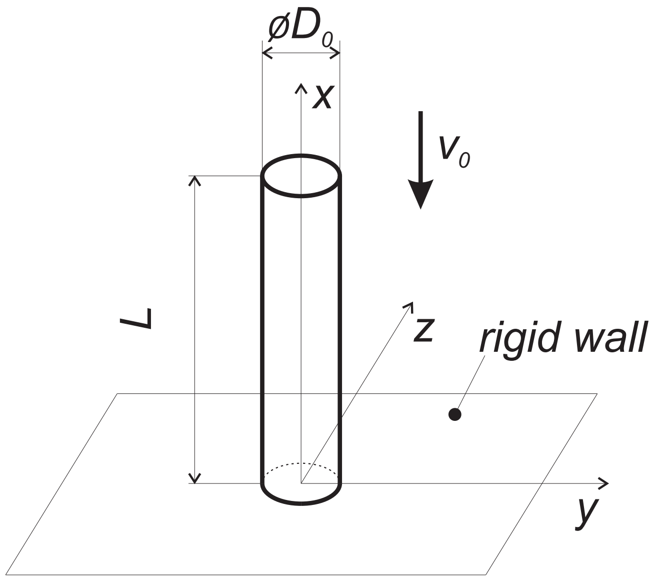


the Park's method



Distributions of σ_{xx}/σ_0 at the time $t = 1.5R/c_L$.

The Taylor test



Geometry: bar radius $R = 3.2$ [mm]
length $L = 32.4$ [mm]
Impact velocity $L = 227.0$ [m/s]
3D problem
Elastic and mass parameters of copper:

$E = 117$ [GPa]

$\nu = 0.35$ [-]

$\rho = 8.93\,[\mathrm{kg/m^3}]$

Simo J_2 finite plasticity theory

Bilinear stress-strain curve

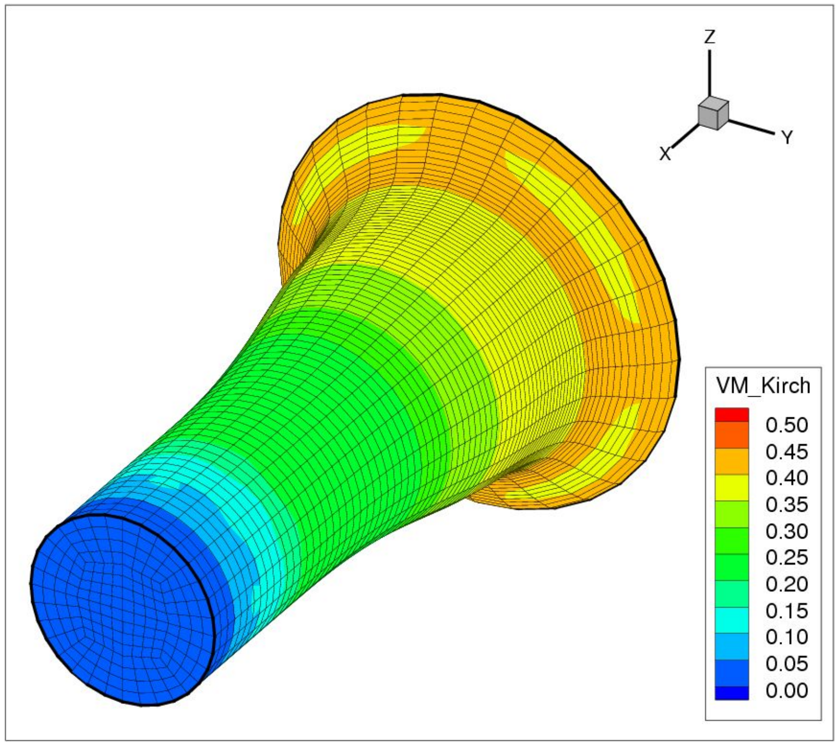
Isotropic hardening

Yield strength $\sigma_Y = 400$ [MPa]

Plastic modulus $E' = 100$ [MPa]

Taylor GI. The use of flat ended projectiles for determining yield stress. I. Theoretical considerations. *Proceedings of the Royal Society A* , **194**, 289–299, 1948.

The Taylor test - dynamic plasticity with strain-rate effect



Distributions of σ_{ekv} at the time $t = 80\ \mu s$.

Integration with local stepping

STEP 1. **Pull-back integration with local stepping**:

1a) Integration by the central difference scheme with the local (elemental) critical time step size Δt_e^{cr} for each finite element at the time $t^{n+cr} = t^n + \Delta t_e^{cr}$

$$(\mathbf{u}_{fs}^{n+cr})_e = \mathbf{u}_e^n + \Delta t_e^{cr} \mathbf{v}_e^n + \frac{1}{2}(\Delta t_e^{cr})^2 \mathbf{a}_e^n \tag{17}$$

$$(\mathbf{a}_{fs}^{n+cr})_e = (\mathbf{M}_e)^{-1} \left[\mathbf{f}_e^{n+cr} - \mathbf{K}_e (\mathbf{u}_{fs}^{n+cr})_e \right] \tag{18}$$

The elemental critical time step size Δt_e^{cr} is set as $\Delta t_e^{cr} = h_e / c_e$ or $\Delta t_e^{cr} = 2/\omega_{max}^e$, where ω_{max}^e is the maximum eigen-angular velocity for the e-th separate finite element.

1b) Pull-back interpolation of local nodal displacement vectors at the time $t^{n+1} = t^n + \Delta t$ with $\alpha = \Delta t / \Delta t_e^{cr}, \quad \beta_1(\alpha) = \frac{1}{6}\alpha\big(1 + 3\alpha - \alpha^2\big), \quad \beta_2(\alpha) = \frac{1}{6}\alpha\big(\alpha^2 - 1\big)$

$$(\mathbf{u}_{fs}^{n+1})_e = \mathbf{u}_e^n + \Delta t_e^{cr} \mathbf{v}_e^n + (\Delta t_e^{cr})^2 \beta_1 \mathbf{a}_e^n + (\Delta t_e^{cr})^2 \beta_2 (\mathbf{a}_{fs}^{n+cr})_e \tag{19}$$

1c) Assembling of local contributions of displacement vector from Step 1b.

$$\mathbf{u}_{fs}^{n+1} = [\mathbf{L}^\intercal \mathbf{L}]^{-1} \mathbf{L}^\intercal (\mathbf{u}_{fs}^{n+1})_e \tag{20}$$

where $\mathbf{L}$ is the assembly Boolean matrix.

STEP 2. Push-forward integration with averaging:

2a) Push-forward predictor of displacement vector at the time $t^{n+1} = t^n + \Delta t$ by the central difference scheme with the time step size Δt.

$$\mathbf{u}_{cd}^{n+1} = \mathbf{u}^n + \Delta t \mathbf{v}^n + \frac{1}{2}\Delta t^2 \mathbf{a}^n \tag{21}$$

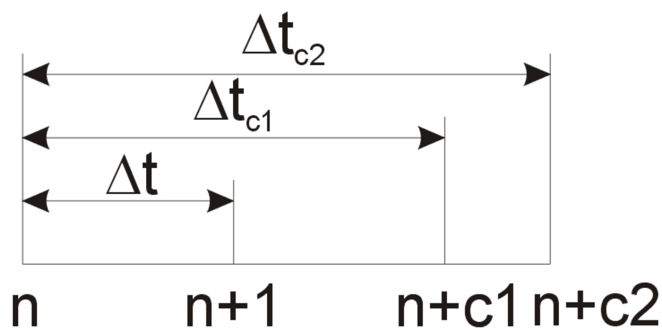
2b) Averaging of the total displacement vectors at the time $t^{n+1} = t^n + \Delta t$ form Steps 1c and 2a for given $\theta = [0, 1]$.

$$\mathbf{u}^{n+1} = \theta \mathbf{u}_{fs}^{n+1} + (1-\theta)\mathbf{u}_{cd}^{n+1} \tag{22}$$

2c) Evaluation of acceleration and velocity nodal vectors at the time $t^{n+1} = t^n + \Delta t$.

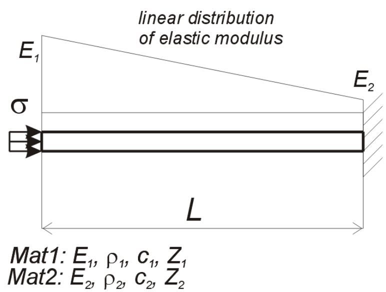
$$\mathbf{a}^{n+1} = (\mathbf{M})^{-1} \left[\mathbf{f}(t^{n+1}) - \mathbf{K}\mathbf{u}^{n+1}\right] \tag{23}$$

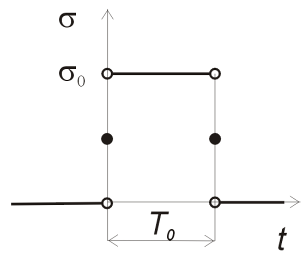
$$\mathbf{v}^{n+1} = \mathbf{v}^n + \frac{1}{2}(\mathbf{a}^n + \mathbf{a}^{n+1}) \tag{24}$$

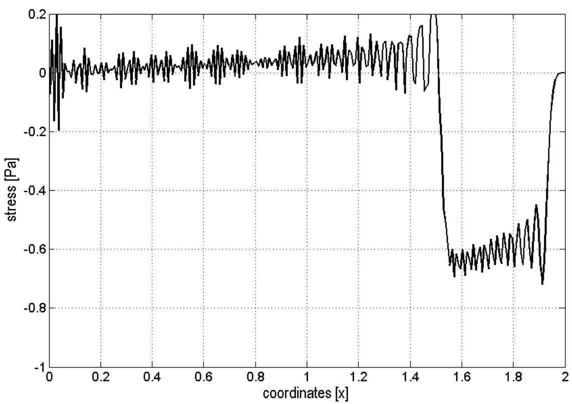


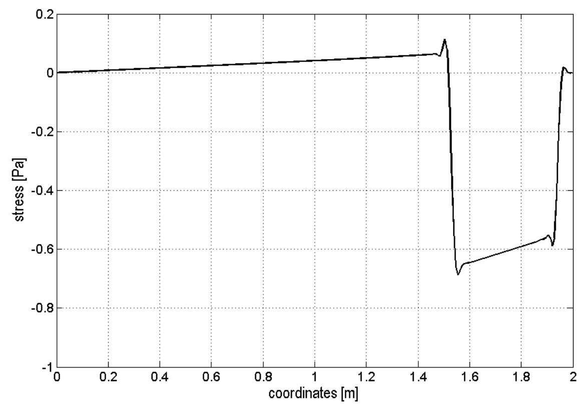
Local stepping scheme via multi-time step method[a]

[a]Kolman, S.S. Cho, J. Gonzalez, K.C. Park, A. Berezovski, V. Adamek, P. Hora. A method with local time stepping for computation of discontinuous wave propagation in heterogeneous solids: Application to one-dimensional problems and unstructured meshes, International Journal for Numerical Methods in Engineering, under progres





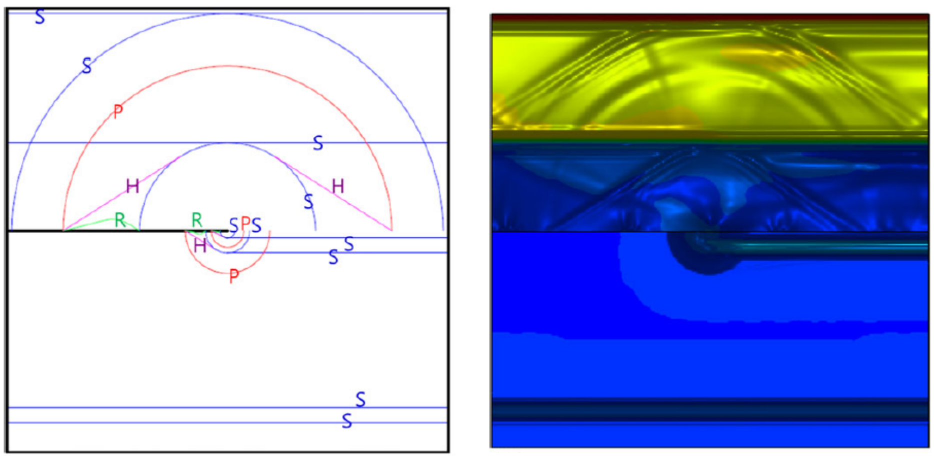




Heterogeneous multi-time step integration for heterogeneous media[a]

[a]S.S. Cho, R. Kolman, J. Gonzalez, K.C. Park. Explicit Multistep Time Integration for Discontinuous Elastic Stress Wave Propagation in Heterogeneous Solids, International Journal for Numerical Methods in Engineering, 2019 Vol. 118, p. 276-302.

For heterogeneous media, the wave speeds are different at each material point.



Accurate wave scheme for anizotropic and heterogeneous media is still an open problem to solve.

15. Contact mechanics with the bipenalty method

Contact mechanics with the bipenalty method

Motivation: Full nonlinear problem - contact impenetrability interfaces are a part of solution with value of normal and tangential forces respecting friction law.

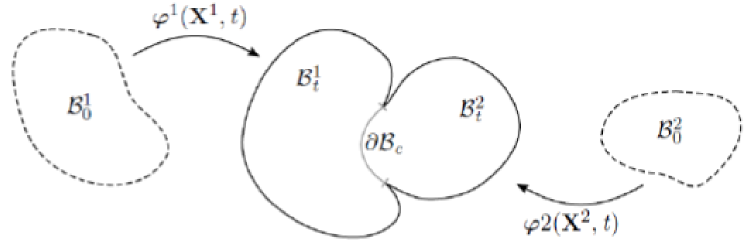
- frictionless contact
- friction contact (Coulomb law, non-Coulomb law - auto-parametric vibration, velocity dependence, split-stick motion)

- Contact of two bodies
- Self-contact

- Static contact
- Dynamic contact - contact-impact problem

Contact Kinematics

Contact of two bodies:

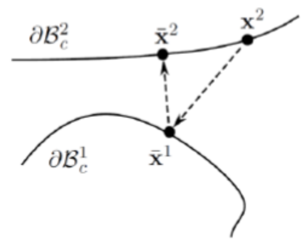


Impenetrability condition

$$\Omega_t^{(i)} \cap \Omega_t^{(k)} = \emptyset$$

where $i \in \{1, 2\}$, and $k = \{1, 2\} \backslash i$.

Close point problems:

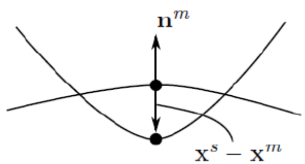


$$\bar{\mathbf{x}}^{(k)} = \arg \min_{\mathbf{x}^{(k)} \in \gamma_c^{(k)}} \left\| \mathbf{x}^{(i)} - \mathbf{x}^{(k)} \right\|$$

Contact Kinematics

Gap function: $g_{\mathsf{N}}^{(i)} := -\left(\mathbf{x}^{(i)} - \mathbf{x}^{(k)}\right) \cdot \bar{\mathbf{n}}^{(k)}$

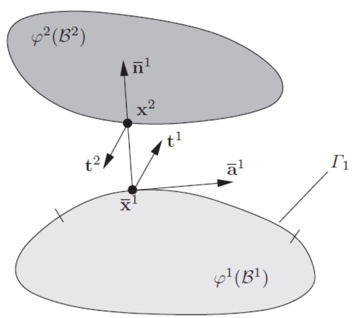
Gap is non-negative: $g_{\mathsf{N}}^{(i)} \geq 0$



The contact traction vector: $\mathbf{t}_{\mathsf{c}}^{(i)} = \boldsymbol{\sigma}^i \mathbf{n}^i$

normal stress (contact pressure): $p_c = \mathbf{t}_{\mathsf{c}}^{(i)} \cdot \mathbf{n}$

contact pressure is compressive: $p_c \leq 0$



Frictionless Contact Boundary Value Problem

Balance of linear momentum

$$\operatorname{div} \boldsymbol{\sigma}^{(i)} + \mathbf{b}^{(i)} = \rho^{(i)} \ddot{\mathbf{u}}^{(i)} \quad \text{in } \Omega^{(i)}$$

Boundary conditions

$$\begin{aligned} \mathbf{u}^{(i)} &= \hat{\mathbf{u}}^{(i)} \quad \text{on } \Gamma_{\mathsf{N}}^{(i)} \\ \boldsymbol{\sigma}^{(i)} \mathbf{n}^{(i)} &= \hat{\mathbf{t}}^{(i)} \quad \text{on } \Gamma_{\mathsf{D}}^{(i)} \\ \boldsymbol{\sigma}^{(i)} \mathbf{n}^{(i)} &= p_{\mathsf{c}}^{(i)} \bar{\mathbf{n}}^{(k)} \quad \text{on } \Gamma_{\mathsf{c}}^{(i)} \end{aligned}$$

Hertz-Signorini-Moreau conditions

$$p_{\mathsf{c}}^{(i)} \leq 0, \qquad g_{\mathsf{N}}^{(i)} \geq 0, \qquad p_{\mathsf{c}}^{(i)} g_{\mathsf{N}}^{(i)} = 0$$

Contact algorithms

Energy balance (principle of virtual work):

$$\delta\mathcal{T} - \delta\mathcal{U} + \delta\mathcal{W} + \delta\mathcal{W}_c = 0$$

Contact virtual work

$$\begin{aligned}
\delta\mathcal{W}_{\mathsf{c}} &= -\int_{\Gamma_{\mathsf{c}}^{(i)}} p_{\mathsf{c}}^{(i)} \bar{\mathbf{n}}^{(k)} \cdot \left(\delta\mathbf{u}^{(i)} - \delta\mathbf{u}^{(k)}\right) \, \mathsf{d}\Gamma^{(i)} \\
&= \int_{\Gamma_{\mathsf{c}}^{(i)}} p_{\mathsf{c}}^{(i)} \delta g_{\mathsf{N}}^{(i)} \, \mathsf{d}\Gamma^{(i)}
\end{aligned}$$

Enforcement of contact constraints

- Penalty method (PM):

$$\mathcal{W}_c = -\int_{\Gamma_c} \frac{1}{2}\epsilon_{\mathsf{N}}(g_{\mathsf{N}}^{(i)})^2 d\Gamma \quad for \quad g_{\mathsf{N}}^{(i)} \geq 0$$

$$p_{\mathsf{c}}^{(i)} = \epsilon_{\mathsf{N}} \left\langle g_{\mathsf{N}}^{(i)} \right\rangle, \qquad \langle x \rangle := \frac{|x| + x}{2}$$

where ϵ_{N} penalty stiffness parameter.

Linearized contact forces: $\mathbf{f}_c = \mathbf{K}_p \mathbf{u}$, where $\mathbf{K}_p$ is the contact stiffness matrix

- Lagrange multiplier method (LMM):

$$\mathcal{W}_c = -\int_{\Gamma_c} \lambda_{\mathsf{N}} g_{\mathsf{N}}^{(i)} d\Gamma \quad for \quad g_{\mathsf{N}}^{(i)} \geq 0$$

$$p_{\mathsf{c}}^{(i)} = \lambda_{\mathsf{N}}^{(i)}$$

Lagrange multiplier - λ_{N}

- Augmented Lagrangian method (ALM):

$$\mathcal{W}_c = -\int_{\Gamma_c} \left(\lambda_{\mathsf{N}} g_{\mathsf{N}}^{(i)} + \frac{1}{2}\epsilon_{\mathsf{N}} (g_{\mathsf{N}}^{(i)})^2 \right) d\Gamma \quad for \quad g_{\mathsf{N}}^{(i)} \geq 0$$

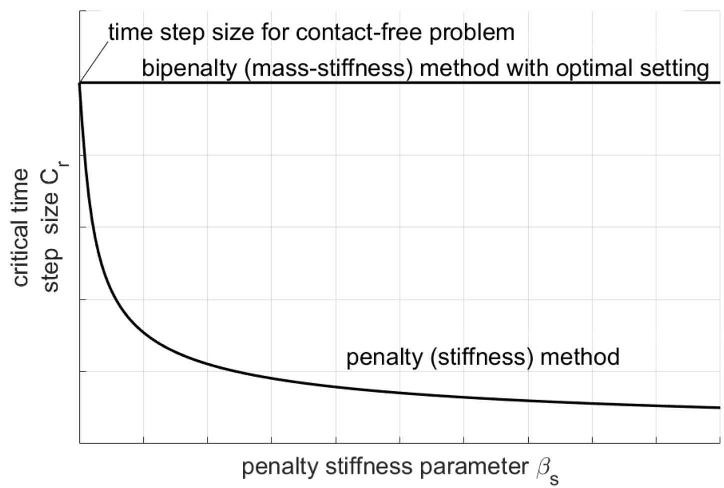
$$p_{\mathsf{c}}^{(i)} = \left\langle \lambda_{\mathsf{N}}^{(i)} + \epsilon_{\mathsf{N}} g_{\mathsf{N}}^{(i)} \right\rangle$$

  - with Uzawa iteration:

$$p_{\mathsf{c}_{\ell+1}}^{(i)} = p_{\mathsf{c}_{\ell}}^{(i)} + \epsilon_{\mathsf{N}} \left\langle g_{\mathsf{N}}^{(i)} \right\rangle$$

Properties of the penalty method in contact problems

The Penalty method - a method for enforcement of contact constraints. It is needed to choice the penalty stiffness parameter.
The penalty method is not the consistent method and Ill-conditioned method.
The solution of contact problems depends on the value of the penalty stiffness parameter ϵ_N.
In this method, the stable time step size is affected by the penalty stiffness parameters ϵ_N.
The penalty method is simply implemented into FEM codes.



Influence of penalty stiffness parameter on stable time step size.

Bipenalty method

Lagrangian functional is enhanced by the bipenalty terms

$$\mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) = \mathcal{T}(\dot{\mathbf{u}}) - \mathcal{U}(\mathbf{u}) + \mathcal{W}(\mathbf{u}) + \mathcal{W}_c(\mathbf{u})$$

Kinetic energy

$$\mathcal{T}(\dot{\mathbf{u}}) = \int_\Omega \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, \mathrm{d}V$$

Strain energy

$$\mathcal{U}(\mathbf{u}) = \int_\Omega \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}V$$

Work of external forces

$$\mathcal{W}(\mathbf{u}) = \int_\Omega \mathbf{u} \cdot \mathbf{b} \, \mathrm{d}V + \int_{\Gamma_\sigma} \mathbf{u} \cdot \mathbf{t} \, \mathrm{d}\Gamma$$

Penalization term associated to the contact interface

$$\mathcal{W}_c(\mathbf{u}) = -\int_{\Gamma_c} \frac{1}{2} \epsilon_{\mathrm{s}} g_{\mathrm{N}}^2 \, \mathrm{d}\Gamma + \int_{\Gamma_c} \frac{1}{2} \epsilon_{\mathrm{m}} \dot{g}_{\mathrm{N}}^2 \, \mathrm{d}\Gamma$$

where g_{N} is the gap function, ϵ_{s} is the stiffness penalty parameter [kg m^{-2} s^{-2}], ϵ_{m} is the mass penalty parameter [kg m^{-2}]

The Hamilton's principle

$$\delta \int_0^T \mathcal{L}(\mathbf{u}, \dot{\mathbf{u}}) \, \mathrm{d}t = 0$$

Variational formulation

$$\int_\Omega \rho \delta \mathbf{u} \cdot \ddot{\mathbf{u}} \, \mathrm{d}V + \int_\Omega \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} \, \mathrm{d}V \; + \; \int_{\Gamma_\mathrm{c}} \delta g_\mathrm{N} (\epsilon_\mathrm{m} \ddot{g}_\mathrm{N} + \epsilon_\mathrm{s} g_\mathrm{N}) \, \mathrm{d}\Gamma$$
$$= \int_\Omega \delta \mathbf{u} \cdot \mathbf{b} \, \mathrm{d}V + \int_{\partial\Omega_\sigma} \delta \mathbf{u} \cdot \mathbf{t} \, \mathrm{d}\Gamma$$

Discretized equation of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{R}_\mathrm{c}(\mathbf{u}, \ddot{\mathbf{u}}) = \mathbf{R}$$

vector of contact forces

$$\mathbf{R}_\mathrm{c}(\mathbf{u}, \ddot{\mathbf{u}}) = \mathbf{M}_\mathrm{p}\ddot{\mathbf{u}} + \mathbf{K}_\mathrm{p}\mathbf{u} + \mathbf{f}_\mathrm{p}$$

where

$$\mathbf{M}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{m} \mathbf{Z}\mathbf{Z}^\mathrm{T} \, \mathrm{d}\Gamma \qquad \mathbf{K}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{s} \mathbf{Z}\mathbf{Z}^\mathrm{T} \, \mathrm{d}\Gamma \qquad \mathbf{f}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{s} \mathbf{Z} g_0 \, \mathrm{d}\Gamma$$

gap function

$$g_\mathrm{N} = \mathbf{Z}^\mathrm{T}\mathbf{u} + g_0$$

For 1D case

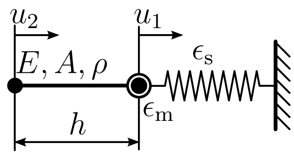
$$\mathbf{Z}^\mathrm{T} = [1, -1]$$

penalty stiffness and mass matrices

$$\mathbf{K}_\mathrm{p} = \epsilon_\mathrm{s} A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{M}_\mathrm{p} = \epsilon_\mathrm{m} A \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Bipenalized Signorini problem

Simple dynamic system with two degrees-of-freedom[18]



$$\frac{EA}{h}\begin{bmatrix} 1+\beta_{\mathrm{s}} & -1 \\ -1 & 1 \end{bmatrix}\mathbf{u} = \omega^2 \frac{\rho A h}{2}\begin{bmatrix} 1+\beta_{\mathrm{m}} & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}$$

where the dimensionless stiffness β_{s} and mass penalty β_{m} are

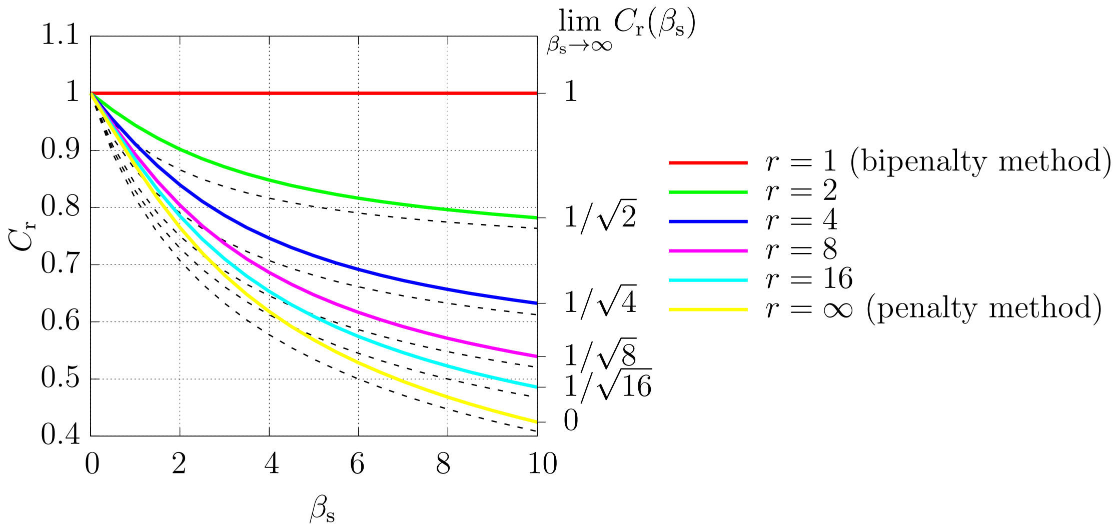
$$\beta_{\mathrm{s}} := \frac{h}{EA}\epsilon_{\mathrm{s}} \qquad \beta_{\mathrm{m}} := \frac{2}{\rho A h}\epsilon_{\mathrm{m}} \qquad r = \frac{1}{2}\frac{\beta_s}{\beta_m}$$

E is the Young's modulus, A is the cross-section, ρ is density.

[18] J. Kopačka, A. Tkachuk, D. Gabriel, R. Kolman, M. Bischoff, J. Plešek. On stability and reflection-transmission analysis of the bipenalty method in contact-impact problems: A one-dimensional, homogeneous case study. *Int. J. Numer. Meth. Engng.* pp 1607-1629, Vol. 113, 2018.

Bipenalized Signorini problem

Bipenalized Signorini problem [19]



General choice of penalized mass matrix [20]

$$\mathbf{M}_p = \frac{1}{\omega_{max}^2} \mathbf{K}_p$$

[19] J. Kopačka, A. Tkachuk, D. Gabriel, R. Kolman, M. Bischoff, J. Plešek. On stability and reflection-transmission analysis of the bipenalty method in contact-impact problems: A one-dimensional, homogeneous case study. *Int. J. Numer. Meth. Engng.* pp 1607-1629, Vol. 113, 2018.

[20] R. Kolman, J. Kopačka, J. Gonzalez, S.S. Cho, K.C. Park. Bi-penalty stabilized technique with predictor-corrector time scheme for contact-impact problems of elastic bars, Mathematics and Computers in Simulation, under preparation of revision

Time integration for the bipenalty method

Central Difference Method in time

$$
\begin{aligned}
\text{Compute nodal accelerations:} \quad \mathbf{a}^{(i)} &= \left(\mathbf{M} + \mathbf{M}_\mathsf{c}\right)^{-1} \left(\mathbf{R}^{(i)} + \mathbf{K}_\mathsf{p}\mathbf{u}^{(i)} - \mathbf{K}\mathbf{u}^{(i)}\right) \\
\text{Update nodal velocities:} \quad \mathbf{v}^{(i+\frac{1}{2})} &= \mathbf{v}^{(i-\frac{1}{2})} + \Delta t \mathbf{a}^{(i)} \\
\text{Update nodal displacements:} \quad \mathbf{u}^{(i+1)} &= \mathbf{u}^{(i)} + \Delta t \mathbf{v}^{(i+\frac{1}{2})}
\end{aligned}
$$

Stability condition:

$$
\Delta t \leq \Delta t_\mathsf{crit} = \frac{2}{\omega_\mathsf{max}}
$$

Time integration for the bipenalty method

Stabilized Central Difference Method with predictor-corrector[21] [22]

Predictor - solution without contact

$$\begin{aligned}
\mathbf{a}_{\text{pre}}^{(i)} &= \mathbf{M}^{-1}\left(\mathbf{R}^{(i)} - \mathbf{F}^{(i)}\right) \\
\mathbf{v}_{\text{pre}}^{\left(i+\frac{1}{2}\right)} &= \mathbf{v}^{\left(i-\frac{1}{2}\right)} + \Delta t \mathbf{a}_{\text{pre}}^{(i)} \\
\mathbf{u}_{\text{pre}}^{(i+1)} &= \mathbf{u}^{(i)} + \mathbf{v}_{\text{pre}}^{\left(i+\frac{1}{2}\right)} \Delta t
\end{aligned}$$

Corrector - only contact

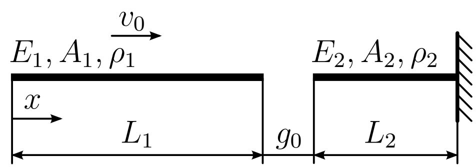
$$\begin{aligned}
\mathbf{a}_{\text{cor}}^{(i)} &= \left(\mathbf{M} + \mathbf{M}_{\text{c}}\right)^{-1}\left(\mathbf{K}_{\text{p}} \mathbf{u}_{pre}^{(i+1)}\right) \\
\mathbf{a}^{(i)} &= \mathbf{a}_{\text{pre}}^{(i)} + \mathbf{a}_{\text{cor}}^{(i)} \\
\mathbf{v}^{\left(i+\frac{1}{2}\right)} &= \mathbf{v}_{\text{pre}}^{\left(i+\frac{1}{2}\right)} + \Delta t \mathbf{a}_{\text{cor}}^{(i)} \\
\mathbf{u}^{(i+1)} &= \mathbf{u}^{(i)} + \mathbf{v}^{\left(i+\frac{1}{2}\right)} \Delta t
\end{aligned}$$

[21] Wu. S.R. A variational principle for dynamic contact with large deformation. in *Comput. Methods Appl. Mech. Engrg.*, pp 2009–2015, Vol. 198 2009.

[22] R. Kolman, J. Kopačka, J. Gonzalez, S.S. Cho, K.C. Park. Bi-penalty stabilized technique with predictor-corrector time scheme for contact-impact problems of elastic bars, Mathematics and Computers in Simulation, under preparation of revision

Impact of two bars - Huněk problem [a]

[a] Huněk, I. On a penalty formulation for contact-impact problems, *Computers & Structures*, **48**(2), 193–203, 1993.

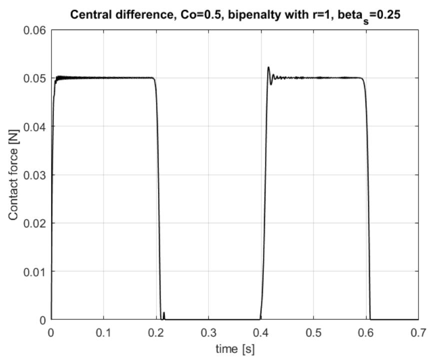


$L_1 = 10\,\mathrm{m}$, $L_2 = 20\,\mathrm{m}$, $v_0 = 0.1\,\mathrm{m \cdot s^{-1}}$, $g_0 = 0\,\mathrm{m}$, $A_1 = A_2 = 1\,\mathrm{m}^2$, $E_1 = E_2 = 100\,\mathrm{Pa}$, $\rho_1 = \rho_2 = 0.01\,\mathrm{kg \cdot m^{-3}}$

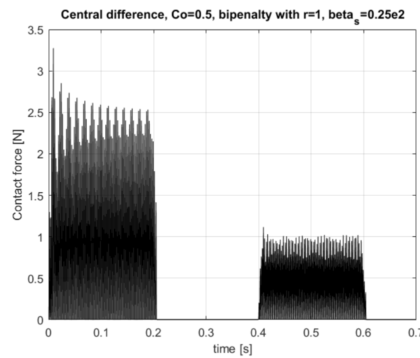
Numerical parameters: number of elements $NELEM_1 = 100$, $NELEM_2 = 200$; Courant number $Co = 0.5$; bipenalty ratio $r = 1$;

chosen stiffness penalty parameters $\beta_s = \{0.25; 0.25e2; 0.25e4; 0.25e8\}$.

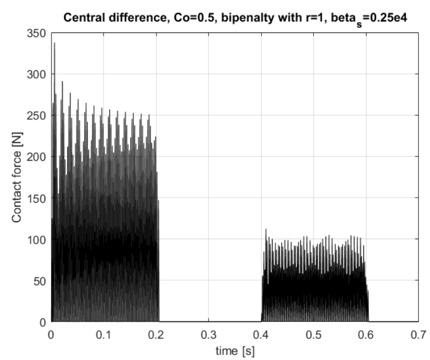
Contact force - CD method with the bipenalty method



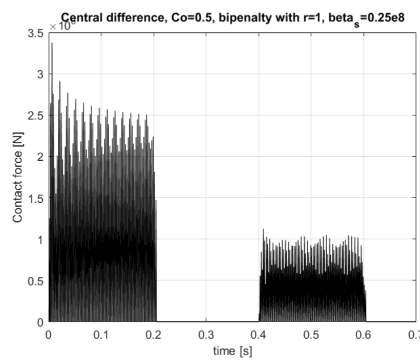
(a) $\beta_\mathrm{s} = 0.25e0$



(b) $\beta_\mathrm{s} = 0.25e2$

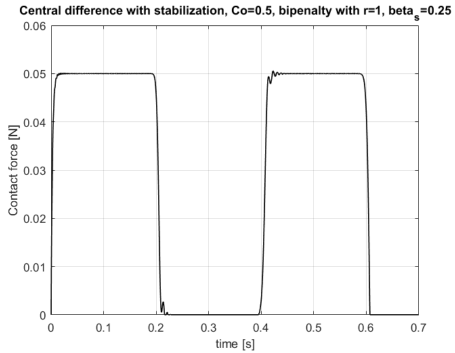


(c) $\beta_\mathrm{s} = 0.25e4$

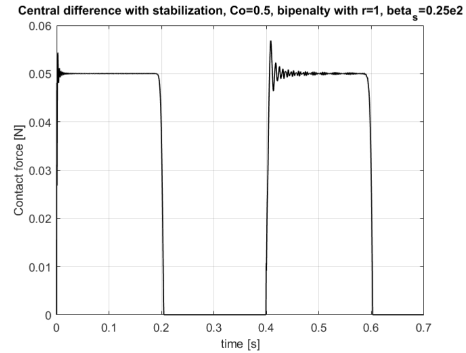


(d) $\beta_\mathrm{s} = 0.25e8$

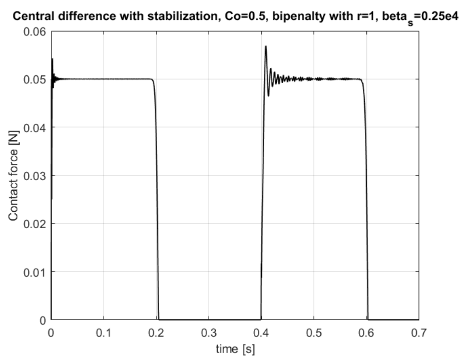
Contact force - Wu method



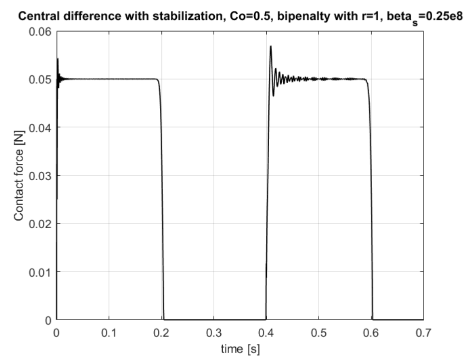
(a) $\beta_{\mathrm{s}} = 0.25e0$



(b) $\beta_{\mathrm{s}} = 0.25e2$



(c) $\beta_{\mathrm{s}} = 0.25e4$



(d) $\beta_{\mathrm{s}} = 0.25e8$

The solution does not depend on the penalty stiffness parameter.

16. Conclusions

- The explicit time integration for real problems is still an open task for study.
- We have studied the properties of explicit time integration in dynamic finite element analysis, wave propagation and contact-impact problems.
- We have suggested new methodologies for accurate modelling of wave propagation problems in solids.
- We have suggested new methodologies for accurate modelling of contact-impact problems of solids.
- Future works - localized versions of the bipenalty stabilization, heterogeneous and asynchronous time integration for heterogeneous and anisotropic media.

Thank you for your attention.